Mathematical analysis of tram tracks

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The article presents a scenario for conducting a lesson in the MIT discipline (Mathematics-Informatics-Engineering; Western term STEM - Science-Technology-Engineering-Mathematics) involving both engineering and the humanities, in particular literature and fine arts (design). Describes how to design a section of railway track for trams and trains.

Key words: function (explicit and implicit), derivative, curvature of a curve, solving equations and their systems, mathematical program SMath.

The professor at the first lecture on calculus shows the textbook and tells the students: "You took 11 years to study the first 17 pages of this book in school, and you will need to study the last 384 pages this semester."
anekdod.ru

The driver of a car, when turning 90 degrees, first smoothly turns the steering wheel to a certain angle, then maintains this angle for some time, forcing the car to move in a circular arc[1], and then again smoothly returns the steering wheel to its original position for straight-line movement, but this time in a different direction, opposite to the previous one. A cyclist or motorcyclist (biker) does approximately the same.

But how does a tram turn? The carriage driver has no steering column: the tram rolls along a rigid rail track. If the rails for turning a tram at an intersection are made in the form of a 90-degree arc of a circle, then this will lead to the fact that the tram at the beginning of the turn and at the end of it will receive quite noticeable lateral shocks, the strength of which depends on the speed of the tram. Imagine that the driver of a car turned the steering wheel extremely sharply while turning a corner. It won't take long to overturn the car! Such jolts of the tram lead to premature wear of the rails and of the tram itself and its wheels. And the passengers won't like it either.

You can, of course, when designing a rail track, simply copy the track of a car at a turn. But you can do it differently and remember the queen of sciences - mathematics, in particular, the mathematical analysis of the function of one argument.

The rails for turning the tram are made in the form of a circular arc, at the ends of which there are inserts, the curvature of which smoothly changes from the value of the reciprocal of the turning

[1] There is an important technical characteristic of such a vehicle, namely the minimum turning radius when the steering wheel is turned to the limit. For passenger cars it is small, for trucks it is larger.

radius to zero. There are many curves that have this property. The simplest of them is a cubic curve. Figure 1 shows two graphs - a graph of the cubic curve itself and a graph of its curvature [2].

$$y(x) := x^3$$

$$\kappa(x) := \frac{\left|\frac{d^2}{dx^2} y(x)\right|}{\sqrt{1+\left(\frac{d}{dx} y(x)\right)^2}^{\,3}}$$

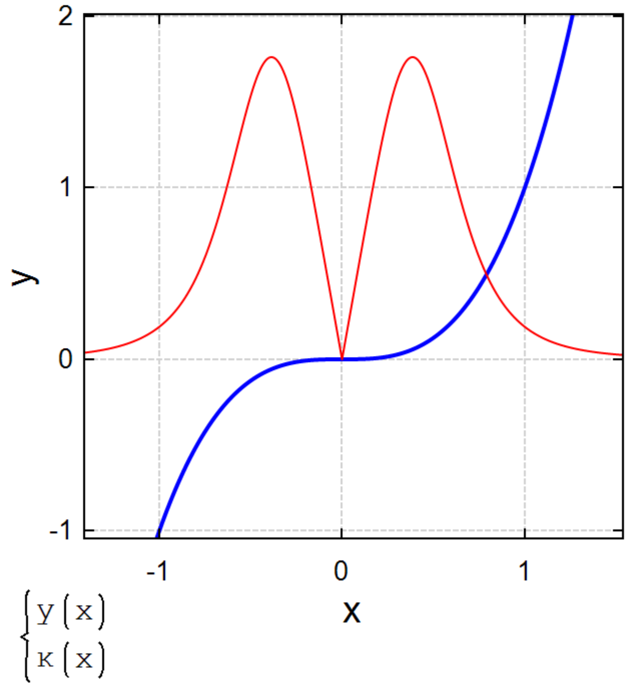


Fig. 1. Cubic curve (blue) and its curvature (red curve)

The formula for the curvature of a curve and its description can be easily found on the Internet using the appropriate search key. These two curves should not really be placed on the same graph, since the corresponding functions have different dimensions along the ordinate axis. For the cubic curve this is the distance, and its curvature is the reciprocal of the distance. But we do not show two separate graphs to save space.

Note. The curvature graph of a cubic is a stylized letter M. It could be a logo (emblem) for a subway under construction in some city! A cubic together with its curvature may be used to profile the rail track at turns not only of trams, but also of metro trains. Such an unusual letter M at the entrance to the subway would once again remind you of the mathematics of the rail track! The most famous logo, in which a certain technical solution is hidden, is that of Citroen. It shows two inverted checkmarks, reminiscent of the fact that the first herringbone gear was used on cars of this particular company.

[2] The calculation was made in the environment of the domestic freely distributed mathematical program SMath - www.smath.com [1].

Figure 2 shows the calculation, in SMath, of a tram leaving a straight track parallel to the abscissa axis ($y = 0$ and $x < 0$), first onto an arc of a cubic curve at $0 < x < x0$, and then onto a circular arc with radius r_o (given as 0.8 in arbitrary units of length[3]). The resulting curve, consisting of a straight line segment and two arcs - a cubic and a circle, has a curvature that changes smoothly and not abruptly.

The solve function built into SMath returns the value of the unknown x_0 (abscissa at the junction of a cubic arc and a circular arc) for a given curvature of the cubic. Thus, the value of the function inverse κ(x) is sought. The solve function is called twice – with and without specifying a range of values. Without a specified range several solutions are given (the given curvature of 1.25 occurs at four points of the cubic parabola), then, with the limited range, the solution we get is 0.2143.

The roots function, also built into SMath, returns the root of a system of two equations - the desired coordinates of the center of the rotation circle. The first equation fixes the fact that the center of the circle lies on the normal to the cubic at the previously found point x_0, y_0. The second equation is the equation of the center of the circle and any of its points.

Кубическая парабола и две её производные

$$y(x) := x^3 \qquad y'(x) := \frac{d}{dx} y(x) \qquad y''(x) := \frac{d}{dx} y'(x)$$

$$\kappa(x) := \frac{\left|y''(x)\right|}{\sqrt{1 + y'(x)^2}^{\,3}} = \frac{\left|6 \cdot x\right|}{\sqrt{1 + 9 \cdot x^4}^{\,3}}$$ Кривизна кривой

$r_o := 0.8$ Исходная величина - радиус поворота

$$\mathrm{solve}\left(\kappa\left(x_0\right) = \frac{1}{r_o}, x_0\right) = \begin{bmatrix} -0.5714 \\ -0.2143 \\ 0.2142 \\ 0.5714 \end{bmatrix}$$

$$x_0 := \mathrm{solve}\left(\kappa\left(x_0\right) = \frac{1}{r_o}, x_0, 0, 0.4\right) = 0.21429$$

$$y_0 := y\left(x_0\right) = 0.009841$$

$$y_н(x) := y\left(x_0\right) - \frac{1}{y'\left(x_0\right)} \cdot \left(x - x_0\right)$$ Уравнение нормали

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} := \mathrm{roots}\left(\begin{bmatrix} y_н\left(x_o\right) = y_o \\ \left(x_o - x_0\right)^2 + \left(y_o - y_0\right)^2 = r_o^{\,2} \end{bmatrix}, \begin{bmatrix} x_o \\ y_o \end{bmatrix}\right) = \begin{bmatrix} 0.1051 \\ 0.8024 \end{bmatrix}$$

Fig. 2. Calculate the tram turn

[3] The SMath mathematical package can also work with dimensional values, which will be useful for checking the correctness of the entered formulas.

In Figure 3, the results of the calculation are displayed graphically. All that remains is to straighten the section of the red cubic curve at x < 0 and remove it at x > x0. This will result in a tram entering a circular section of track. All that remains is to calculate the second rail!

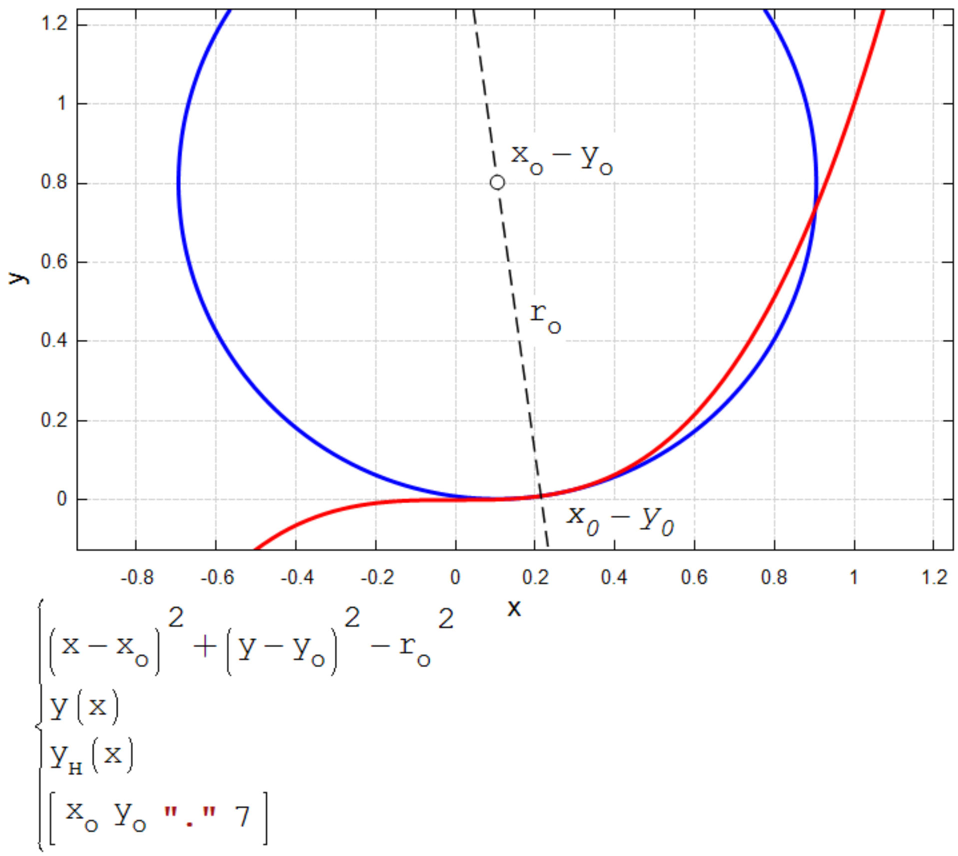


Fig. 3. Draw a tram turn

Figure 3a shows an enlarged section of the tram path at the exit from a straight line (black), to a round one (blue), through a cubic parabola (red). Also shown is the normal (a dotted line passing through the center of the circle), common to both the cubic parabola and the circle. Below the first graph is the second one, which characterizes the change in the curvature of the turn.

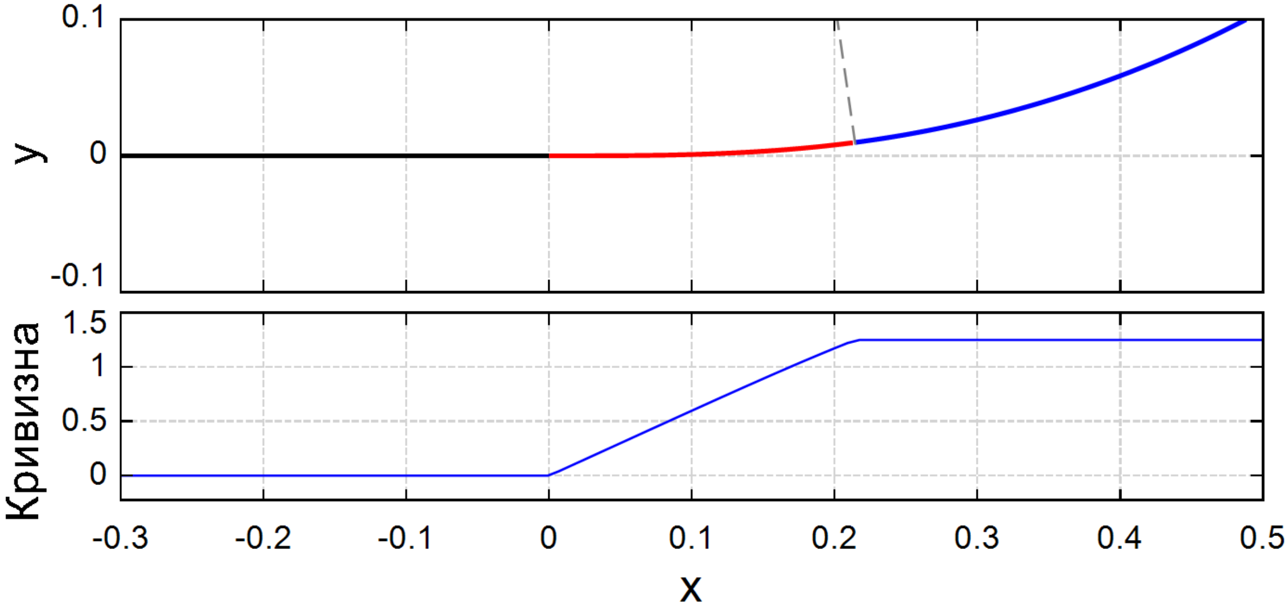


Fig. 3a. Derailment curve of a tram from a straight to a round section with its curvature

By the way, old trams, when turning, “cleared away unwary pedestrians” not only with bells, but also with a rather loud squeal of wheels. This was due to the fact that these trams had not just wheels, but wheel pairs like trains. Modern low-floor trams have separate wheels. In a wheel pair, the angular speed of rotation of the left and right wheels is always the same. And it should be different when turning. Hence the squeal of wheels slipping! You can hardly hear such a squeal on metro trains because their turning radii are much larger than those on trams.

Figure 4 shows a photo of a railway transport test site in the village of Shcherbinka near Moscow. This is an oval approximately 6 kilometers long with two straight sections. Now we've reached the oval! And then there were all circles and parabolas.

Fig. 4. Railway transport site

A closed curve consisting of two equal parallel segments and two semicircles connecting them in the West is called a Stadium [4]. It cannot be called an oval, not only because it is not completely smooth - it has breaks in four places. It will become smooth if four segments of a cubic curve are inserted into it, the calculation of which we described above (see Fig. 2 and 3). In addition, an oval is a convex closed smooth curve that any straight line can intersect at most two points. But the closed Stadium curve does not meet this condition - its "sides" have an infinite number of points along the straight lines.

Note. For railway transport, especially for high-speed lines, where the danger of derailment of rolling stock is much higher than that of a tram, the transition curve is not a cubic, which is easy to calculate, but a clothoid (Euler spiral - see Fig. 5), which has curvature varies linearly with the length of the spiral. Integrals also have a name - these are Fresnel integrals, which the SMath package perfectly understood and built a so-called parametric graph with the parameter t. If this parameter tends to infinity, then the Euler spiral will collapse to a point at which the curvature is infinite.

[4] A classic stadium with two semicircles and two straight segments is called the Bunimovich stadium. This curve is used in the study of so-called dynamic billiards [4], in which balls (material points) are "chased from side to side," bouncing off the sides perfectly elastically.

$$x(t) := \int_0^t \cos\left(u^2\right) d\,u \quad y(t) := \int_0^t \sin\left(u^2\right) d\,u$$

$$t := [0, 0.001 .. 6]$$

$$clothoid := \text{augment}\left(\overrightarrow{x(t)}, \overrightarrow{y(t)}\right)$$

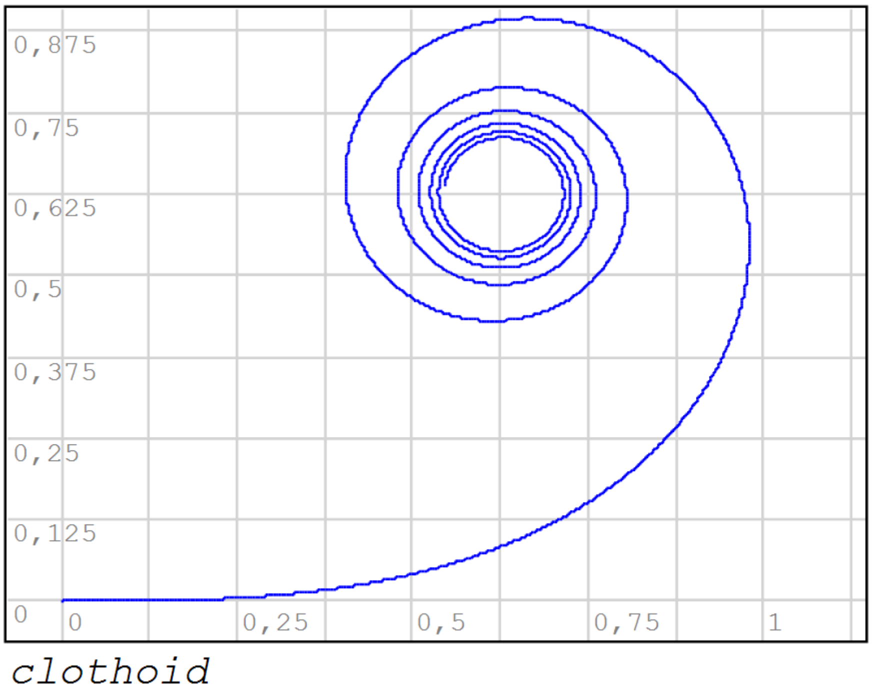


Fig. 5. Clotoid built in SMath

A closed curve for a railway site may have a more interesting shape. Let's talk about that!

The most famous oval is the ellipse. It is this, and not just an oval, that is drawn by the PaintBrush graphic editor - see Chapter 1 in [1]. Let's use SMath to construct an ellipse and two other lesser-known, but nevertheless very interesting ovals related to the concept of average value.

To begin with, we will slightly change the definition of an ellipse.

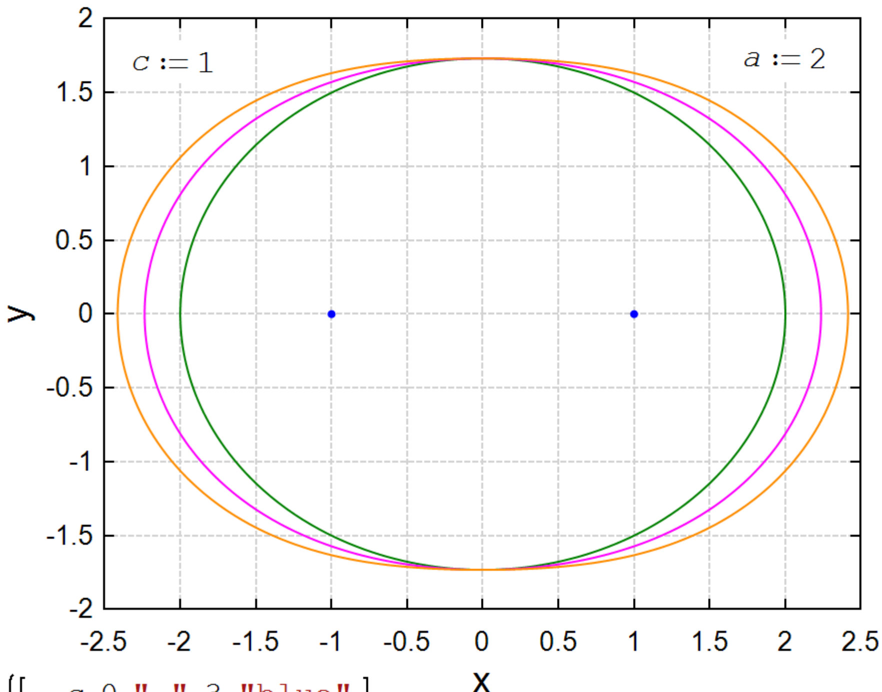
An ellipse is a geometric locus of points on a plane for which the arithmetic mean of the distances to two foci is constant and equal to a given value. If in this definition the words “arithmetic mean” are replaced with the word “sum” (twice the arithmetic mean), then we will return to the old classical definition of an ellipse. If the arithmetic mean (mean) is replaced by the geometric mean (gmean) or the harmonic mean (hmean), then we get the so-called Cassini oval or Cayley oval. Figures 6-10 show the construction of these two ovals together with an ellipse for the same values of the parameter c (the distance from the origin to the foci) and for different values of the parameter a (the value of the arithmetic mean, geometric mean and harmonic mean).

$$mean(a, b) := \frac{a+b}{2} \qquad gmean(a, b) := \sqrt{a \cdot b} \qquad hmean(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

$$Ellipse(x, y, a) := mean\left(\sqrt{(x-c)^2 + y^2}, \sqrt{(x+c)^2 + y^2}\right) - a$$

$$Cassini(x, y, a) := gmean\left(\sqrt{(x-c)^2 + y^2}, \sqrt{(x+c)^2 + y^2}\right) - a$$

$$Caley(x, y, a) := hmean\left(\sqrt{(x-c)^2 + y^2}, \sqrt{(x+c)^2 + y^2}\right) - a$$



$$\begin{cases} [-c \ 0 \ \text{"."} \ 3 \ \text{"blue"}] \\ [c \ 0 \ \text{"."} \ 3 \ \text{"blue"}] \\ \mathrm{Ellipse}(x, y, a) \\ \mathrm{Cassini}(x, y, a) \\ \mathrm{Caley}(x, y, a) \end{cases}$$

Fig. 6. Construction of the ellipse (green), Cassini oval (pink) and Cayley oval (brown, a = 2)

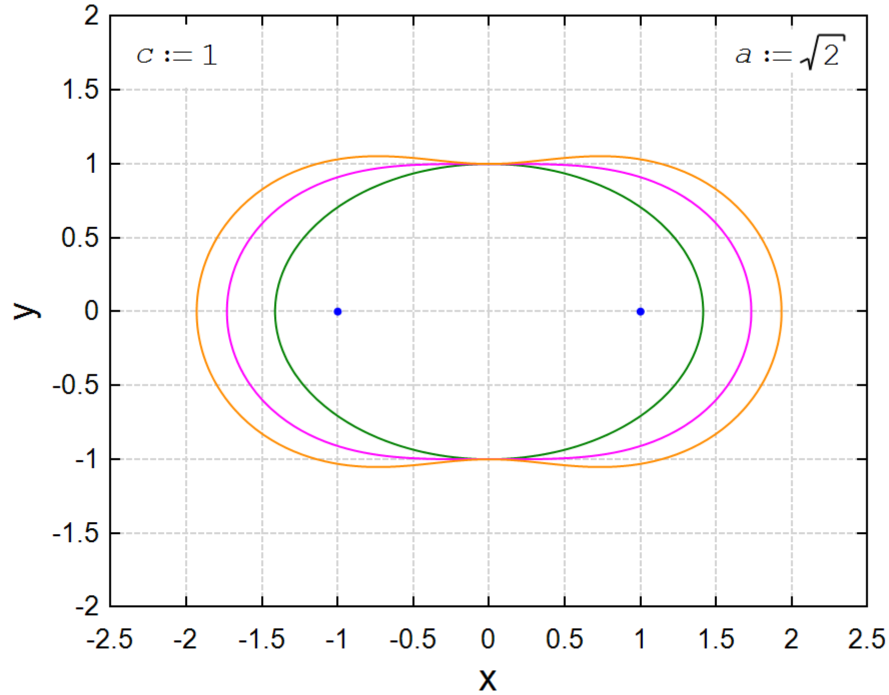


Fig. 7. Ellipse, Cassini Oval and Cayley Oval

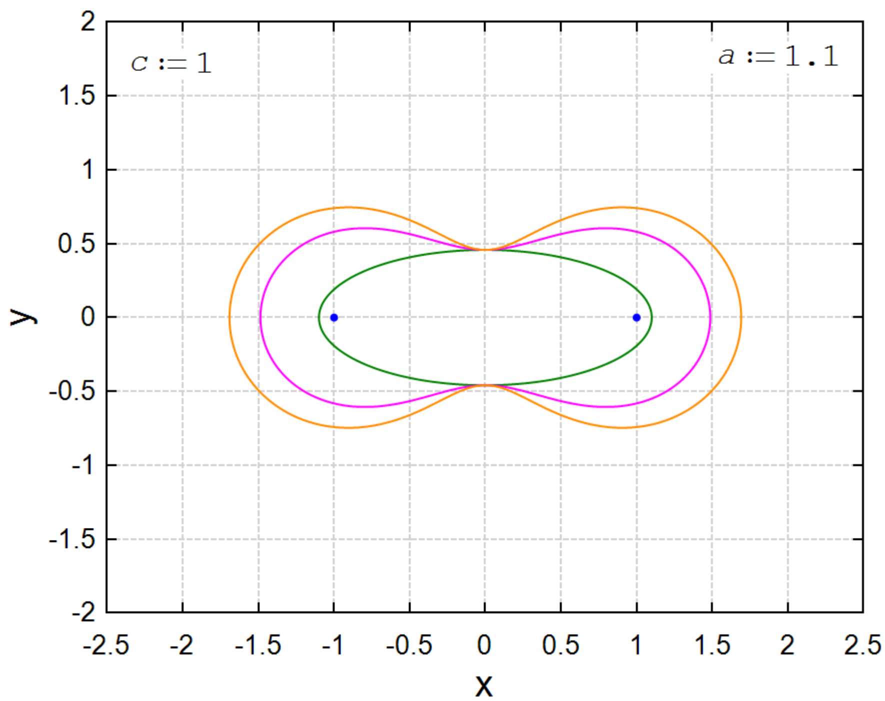


Fig. 8. Ellipse, Cassini Oval and Cayley Oval

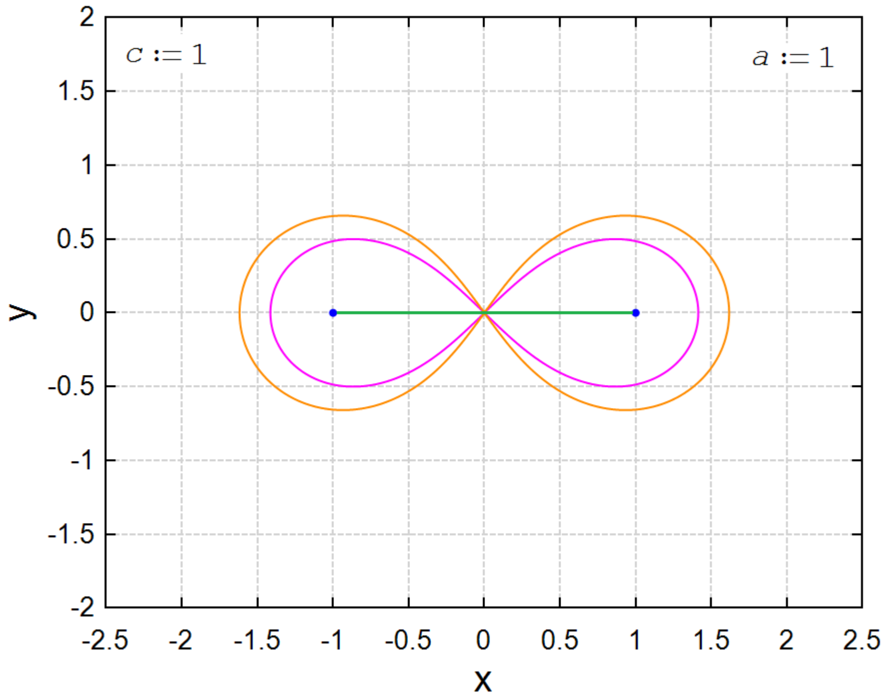


Fig. 9. Ellipse degenerated into a straight line segment, Cassini oval (Bernoulli lemniscate) and Cayley oval

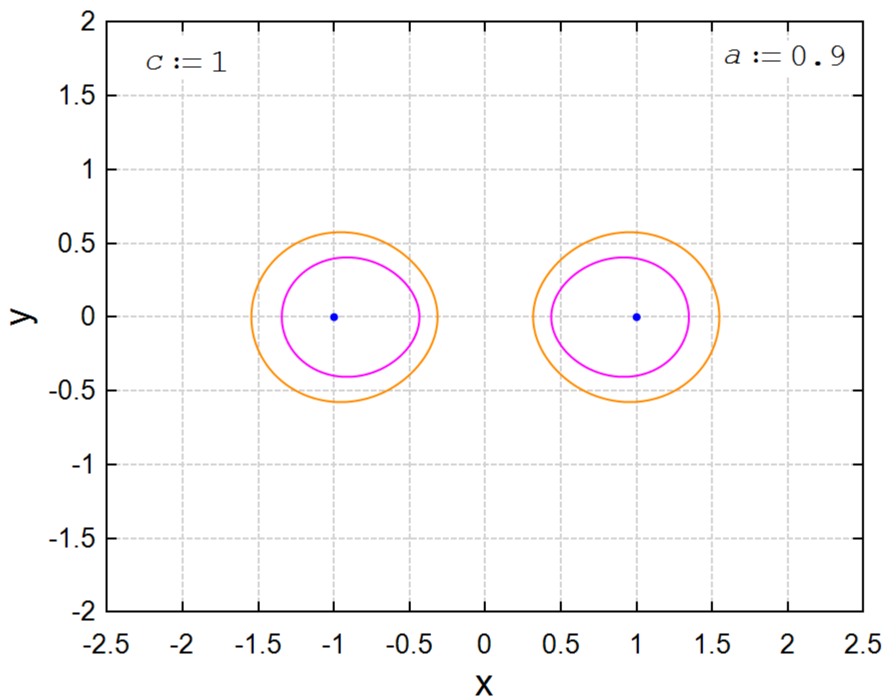


Fig. 10. Two Cassini ovals and two Cayley ovals (the ellipse is invisible, it is imaginary)

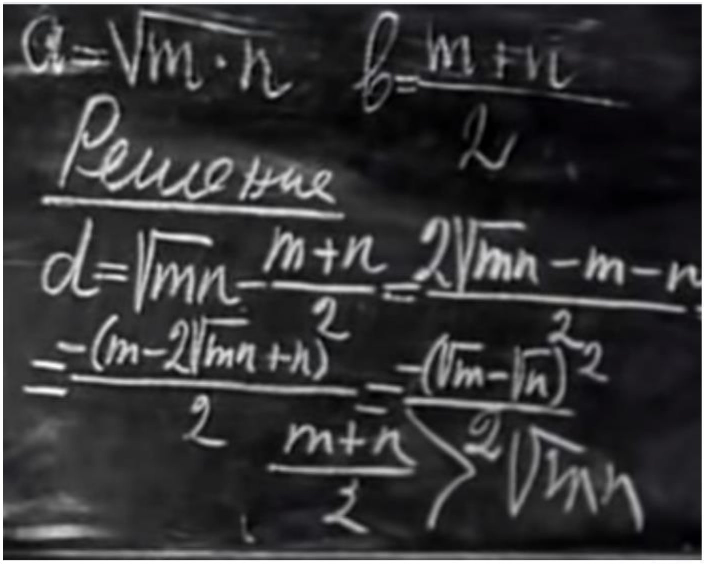
Cinematic retreat.

In the Soviet feature (not popular science!) film “Hearts of Four” there is a scene at the blackboard (Fig. 11a), when two heroes of the film (two hearts out of four) discuss the arithmetic mean and the geometric mean.

Fig. 11a. Still from the film “Hearts of Four”

The heroine of the film, Galina Murashova (Associate Professor of Mathematics at Moscow State University - played by Valentina Serova), believes that she is the “arithmetic mean,” and her younger sister-student Shurochka (Lyudmila Tselikovskaya) is the “geometric mean.” The film will prove that the arithmetic mean is greater than the geometric mean. Thus, Galina falls in love with senior lieutenant Pyotr Kolchin (Evgeny Samoilov - a handsome man - he gives the proof at the blackboard), she herself becomes partial to him and wins this gentleman away from her younger sister (third heart). The character of the fourth heart - the heart of Gleb Zavartsev (scientist-biologist - artist Pavel Springfeld), first looks after Galina, and then Shurochka. This is a love quadrangle!

Figure 11b shows the final screenshot of this scene, and below it is one of the correct proofs.



$$\frac{a+b}{2} \geqslant \sqrt{ab} \;\Leftrightarrow\; (a+b)^2 \geqslant 4ab \;\Leftrightarrow\; (a-b)^2 \geqslant 0$$

Fig. 11b. Still from the film "Hearts of Four"

Figures 6-9 with our ellipses and ovals can be considered another geometric interpretation of this "love proof": the ellipse is always inside the Cassini oval. By the way, the Cayley oval is always larger than both the Cassini oval and the ellipse. Hence the conclusion: The harmonic mean is less than or equal to (not greater than) the geometric mean.

To be fair, we note that the solution with chalk on the blackboard in Fig. 10b, and our statements just given are not entirely correct: we need to write and say "greater than or equal to," and not just "greater than." In Figures 6-9, our three closed curves converge in two places: our three average values are equal to each other when the two numbers being compared are equal. But Galina does not notice this mathematical error - she is already head over heels in love with Peter. These are the miracles mathematics works!

Among the Cassini ovals there are two named ones. These are the Bernoulli lemniscate (pink curve in the form of an infinity sign in Fig. 9) and Tolstoy's oval (pink curve in the form of an oval in Fig. 7). The first closed curve is well known. The second curve is just trying to take its place in the "pantheon of named objects of mathematics" [2, 3]. Tolstoy's oval has no "waist," so to speak. The curvature of the Cassini oval at x = 0 will be zero - see fig. 11, where the explicit function of this oval, obtained by solving the Cassini oval equation (implicit function), is used with respect to the unknown value y. The analytical solution[5] is found through the use of the Maple mathematical program, which is called from an SMath plug-in. Four solutions are given, of which only the first is used. The Cassini oval is a curve of the fourth, not the second order (ellipse). I wonder what order the curve of the Cayley oval is?

[5] Figure 2 shows the numerical solutions

$$Cassini\left(x, y, a, c\right) := \sqrt{\sqrt{\left(x-c\right)^{2}+y^{2}}\cdot\sqrt{\left(x+c\right)^{2}+y^{2}}} - a$$

$$\begin{bmatrix} Cassini\left(x, a, c\right) \\ No \\ No \\ No \end{bmatrix} := \mathrm{maple}\left(\mathrm{solve}\left(Cassini\left(x, y, a, c\right) = 0, y\right)\right) = \begin{bmatrix} \sqrt{-\left(c^{2}+x^{2}-\sqrt[2]{4\cdot x^{2}\cdot c^{2}+a^{4}}\right)} \\ -\sqrt{-\left(c^{2}+x^{2}-\sqrt[2]{4\cdot x^{2}\cdot c^{2}+a^{4}}\right)} \\ \sqrt{-\left(c^{2}+x^{2}+\sqrt[2]{4\cdot x^{2}\cdot c^{2}+a^{4}}\right)} \\ -\sqrt{-\left(c^{2}+x^{2}+\sqrt[2]{4\cdot x^{2}\cdot c^{2}+a^{4}}\right)} \end{bmatrix}$$

$$\kappa\left(x, a, c\right) := \frac{\left|\frac{d^{2}}{dx^{2}} Cassini\left(x, a, c\right)\right|}{\sqrt{1+\left(\frac{d}{dx} Cassini\left(x, a, c\right)\right)^{2}}^{\,3}}$$

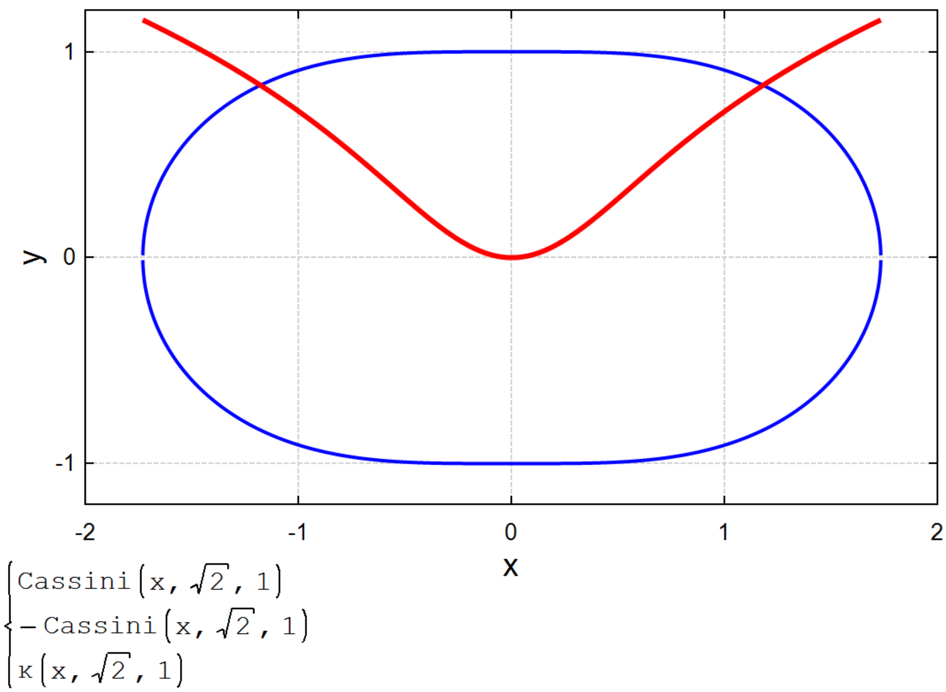


Fig. 12. Tolstoy’s oval and its curvature

The image shown in Fig. 12 may also become a logo, not for the metro (see Fig. 1), but for a plant that produces rolling stock for railways. The UAZ car has a similar emblem, but it depicts not an oval, but a circle and a “tick” in the form of Chekhov’s seagull. In our Fig. 12 it turned out to be a kind of Tolstoy's seagull!

We read from Tolstoy in “Anna Karenina”: “*The races were supposed to take place on a large four-mile elliptical circle in front of the gazebo.*” And one more thing: “*Above the chair hung an oval portrait of Anna, beautifully made by the famous artist, in a gold frame*.” Tolstoy was an artillery officer, and he should have known what an ellipse was and what an oval was. But why did he call one closed curve an ellipse and the second an oval? Rather, it’s the other way around: the frame of the picture is an ellipse, and the hippodrome is an oval, or rather, a stadium and not just a simple one, but a Tolstoy stadium.

The heroes of this most famous ladies' novel - Anna Karenina and Alexey Vronsky meet on the railway - on Nikolaevsky (now Leningradsky) in Moscow. Anna's life is tragically interrupted, again on the railway. Konstantin Levin (Tolstoy's alter ego) talks a lot about the benefits and harms of railways [2].

If the Tolstoy oval (see Fig. 12) is cut vertically into two halves, moved left and right, and the ends of the arcs are connected with straight segments, then we get the Tolstoy stadium (Fig. 13) - the most suitable closed curve for both a hippodrome and for a railway test site.

$$X_F := \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad Y_F := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad N := \mathrm{rows}\left(X_F\right) = 2$$

$$Cassini\left(x, y\right) := 2 - \prod_{i=1}^{N} \sqrt{\left(x - X_{F_i}\right)^2 + \left(y - Y_{F_i}\right)^2}$$

$$\begin{aligned} Stadium\left(x, y\right) := \ & \mathrm{if}\ \ x < -1 \\ & \quad Cassini\left(x+1, y\right) \\ & \mathrm{else} \\ & \quad \mathrm{if}\ \ x > 1 \\ & \quad\quad Cassini\left(x-1, y\right) \end{aligned}$$

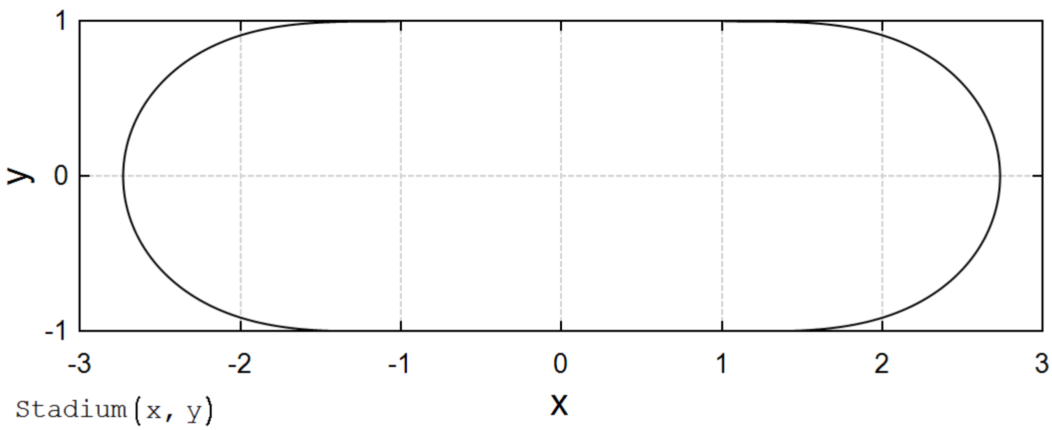


Fig. 13. Tolstoy Stadium

In Figure 13 we see a calculation of the Cassini oval in a form that allows us to rely not just on two, but on a larger number of foci. Reader, build them. To do this, it is enough just to increase the length of the XF and YF vectors and enter the coordinates of the additional foci. Another more complex task is to design a stadium based on the Cayley Oval.

By the way, a huge number of such ovals can be invented, based on Kolmogorov's concept of the average [5].

Conclusion.

In 2028, the 200th anniversary of the birth of Leo Nikolaevich Tolstoy will be celebrated.[6] On this date, speaking in bureaucratic language, "many anniversary events will be timed." It would be great to build a

[6] There is a half-joking test related to the division of people into techies and humanists, into physicists and lyricists. Humanists remember Tolstoy's birth date (1828) through the number e (2.718281828), and techies remember the number e through the year of Tolstoy's birth. Or vice versa. Also, through geometry - "the heroine of this article" - just

hippodrome or stadium named after Leo Tolstoy for this anniversary with a running track in the form of the Tolstoy stadium. Another relevant project. On Lev Tolstoy Street in some city, in Tula, for example, one could build a new tram line with turns in the form of an arc not of a cubic curve, but of a Tolstoy oval. And surround all this with an appropriate information field about Leo Tolstoy and mathematics. Here is a quote from the autobiographical story "Youth": "*I am preparing for the Faculty of Mathematics, and this choice, to tell the truth, was made by me solely because the words: sines, tangents, differentials, integrals, etc., I really like*." And another quote from this story: "*I stood near the window <...> and solved some long algebraic equation on the black board. In one hand I held Francoeur's torn soft "Algebra", in the other a small piece of chalk, with which I had already stained both hands and my face...*" (see Fig. 11a).

If we talk about a specific MIT[7] lesson (Mathematics-Informatics-Engineering), then schoolchildren and students can correctly calculate the profile of a model railway, then print rails with sleepers on a 3D printer and run a winding engine on it, which will be without lateral pushes, to ride around the Tolstoy Stadium.

During his student years, the first author participated in a theatrical production of this novel by Tolstoy, where the characters were Anna Karenina and the Steam Locomotive (the main characters), as well as Carriages, Rails and Sleepers (extras). Let's remember the abbreviation STEM - student theater of variety miniatures. Nowadays, this abbreviation has received a new meaning: STEM (Science-Technology-Engineering-Mathematics) or MIT in Russian adaptation.

Further:

The reader can view this article vertically or horizontally. This does not mean the position of the reader, but the position of the article, which can be opened either in a paper magazine or on a tablet lying on the table. This is the horizontal position of the article. It will be in a vertical position when it is opened on a personal computer monitor or on a laptop (almost horizontal position). Let's return to Figure 2.

In a horizontal position, these are tram tracks or railways. And in a vertical position it's... a loop of a "Roller Coaster" attraction[8], where it is also necessary to profile the rails for the cabins (cars) so that there are no strong vertical shocks.

But here is an idea that is even worth patenting!

A loop of such rides can be made with a Tolstoy Stadium profile - see Figure 13. On such a section, firstly, there will be no sharp shocks, and secondly, the cabins of these slides on the upper section of the loop will move in a straight line, slowing down. People's hearts will skip a beat because they are about to stop and hang upside down. But everything ends well - the cabins fall down along the arc of the Tolstoy oval... It will only be necessary to determine the optimal length of the rectilinear inserts into the Tolstoy oval (Fig. 8 and 13), making them the stadium of the same name.

The Tolstoy Stadium cannot be placed vertically ("on the butt"). We need a hybrid of a rollercoaster and a diving coaster!

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remember 6 more digits of the number e: 2.718281828459045. To do this, just remember the angles of an isosceles right triangle.

[7] Or MIIT - Mathematics-Informatics-Art-Technology. MIIT, by the way, is also the Moscow Institute of Transport Engineers - the hero of our story.

[8] In some European countries they are called Russian roller coasters.

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