⊞— bvp

Boundary Values Problem

appVersion(4) = "0.99.7921.69"

$lbvp_2$ $\quad lbvp_2(f(x), ab, M, N)$ solves the linear ODE in $ab = [a \;\; b]$

$$y'' + p \cdot y' + q \cdot y = r$$

subject to the boundary conditions

$$\begin{cases} \alpha_1 \cdot y(a) + \beta_1 \cdot y'(a) = c_1 \\ \alpha_2 \cdot y(b) + \beta_2 \cdot y'(b) = c_2 \end{cases}$$

f is such that $f(x) = [p(x) \;\; q(x) \;\; r(x)]$ and $M = \begin{bmatrix} \alpha1 & \beta1 & c1 \\ \alpha2 & \beta2 & c2 \end{bmatrix}$

$sb\_lbvp_2(bvp, y, x, N)$ for use in the Solve block

bvp_2 $\quad bvp_2(\varphi(x, y, y'), x, 0, M, N, \varepsilon)$ solves the non linear ODE in $x = [a \;\; b]$

$$y'' = \varphi(x, y, y')$$

subject to the the same boundary conditions, with Yo as guess for the solution with dimension N+1. If Yo=0, bvp.2 try with a line between f(a) and f(b) as guess.
ε is used as the tolerance for a Newton solver.

$sb\_bvp_2(bvp, y, x, [a \;\; b], N, Yo, \varepsilon)$ for use in the Solve block

⊟— lbvp example

Example $\quad [a \;\; b] := [0 \;\; 5] \qquad h := x \cdot \cos(x) \qquad h' := \frac{d}{dx} h$

Numeric solution

$$y'' + 2 \cdot h \cdot y' + \left(h^2 + h'\right) \cdot y = 0$$

$$y(a) + 3 \cdot y'(a) = 0.1$$
$$y'(b) = -6$$

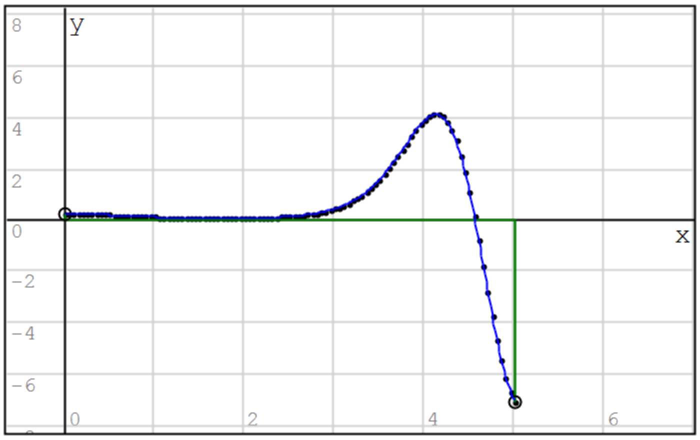
$sol := sb\_lbvp_2(y, x, [a \;\; b], 100)$

Analytic solution

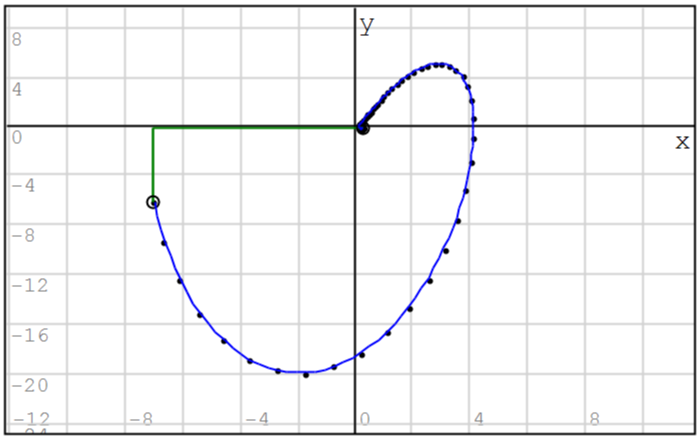
$$\psi(x) := (A + B \cdot x) \cdot e^{-(x \cdot \sin(x) + \cos(x))}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} := \text{roots}\left(\begin{bmatrix} \psi(a) + 3 \cdot \psi'(a) = 0.1 \\ \psi'(b) = -6 \end{bmatrix}, \begin{bmatrix} A \\ B \end{bmatrix}\right)$$

$Err = 0.16$



Plot("XY", sol)



Plot("YY'", sol)

⊟— lbvp example

Example

Numeric solution

$$y'' + 3\cdot y' - 4\cdot y = 0$$

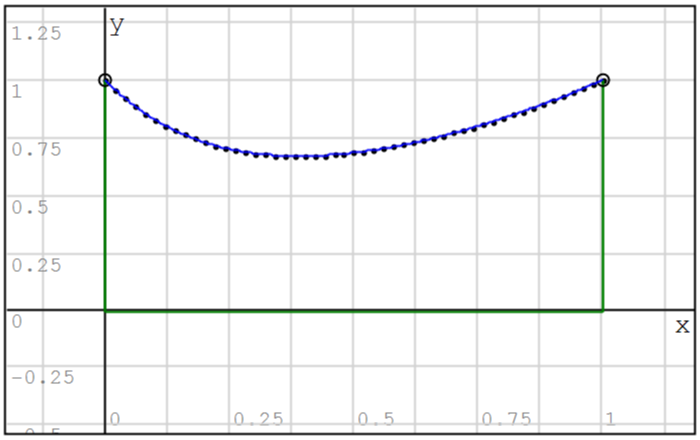
$$y(0) = 1 \qquad y(1) = 1$$

$sol := sb\_lbvp_2(y, x, [0\ 1], 50)$

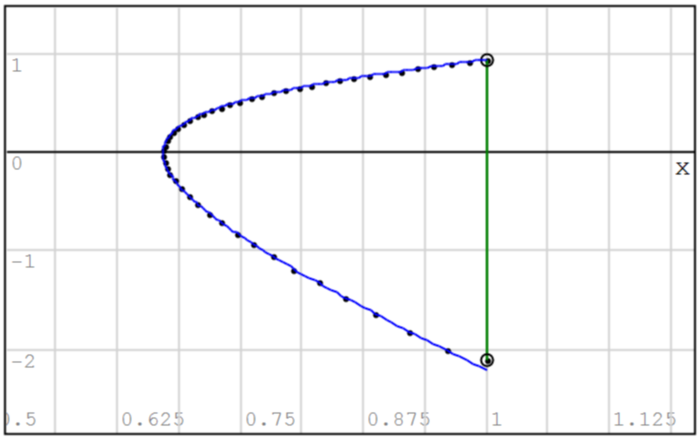
Analytic solution

$$\psi(x) := \frac{\left(e^{(4-4\cdot x)} + e^{x} + e^{(x+1)} + e^{(x+2)} + e^{(x+3)}\right)}{\left(1 + e + e^{2} + e^{3} + e^{4}\right)}$$

$Err = 0$



Plot ("XY", sol)



Plot ("YY'", sol)

⊟— lbvp example

Example

Numeric solution

$$y'' + 3\cdot y' + 6\cdot y = 5 \qquad y(1) = 3$$

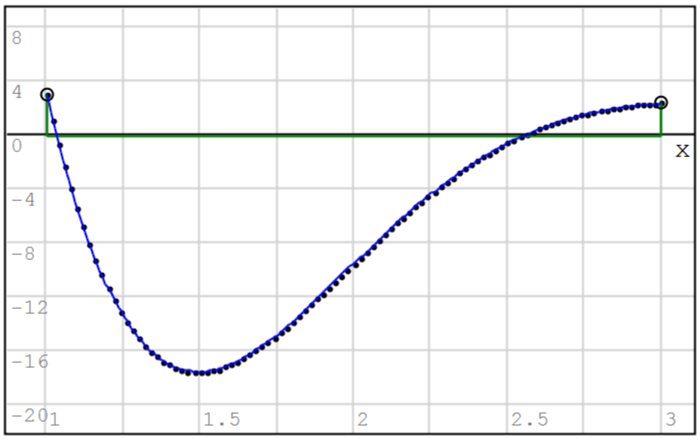
$$y(3) + 2\cdot y'(3) = 5$$

$sol := sb\_lbvp_2(y, x, [1\ 3], 100)$

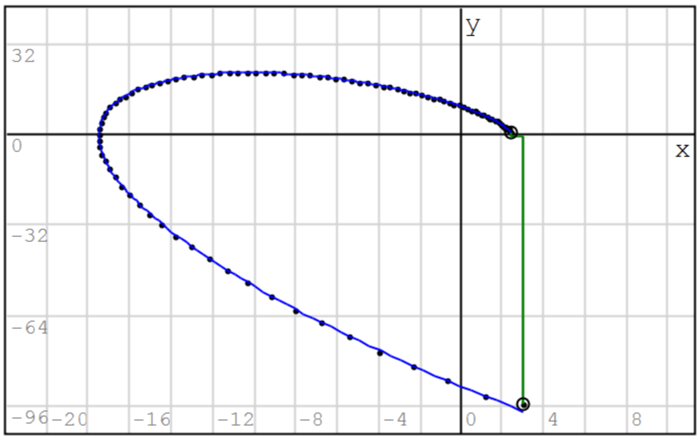
Analytic solution

$\psi(x) := 0.0696737\cdot e^{-1.5\cdot x}\cdot\left(11.9605\cdot e^{1.5\cdot x} + 1243.5\cdot \sin(1.93649\cdot x) + 2857.67\cdot \cos(1\right.$

$Err = 0.21$



Plot ("XY", sol)



Plot ("YY'", sol)

⊟— lbvp example

Example

$p := 10^{-5} \qquad [a\ b] := [-0.1\ 0.1]$

Analytic solution

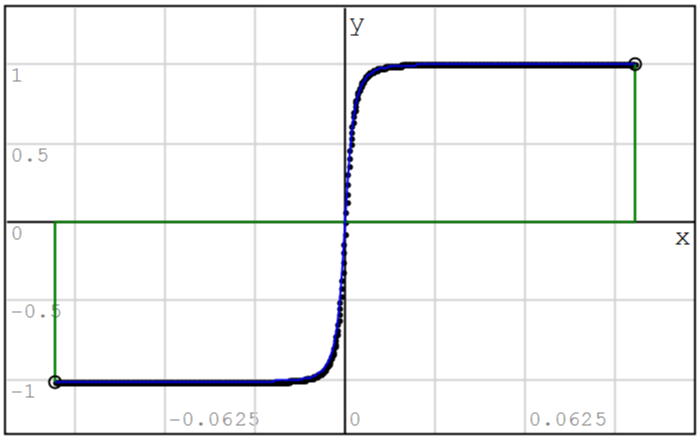
$$\psi(x) := \frac{x}{\sqrt{p + x^{2}}}$$

Numeric solution

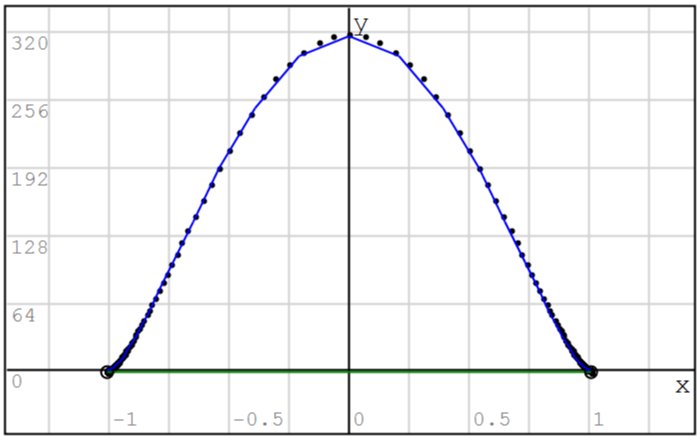
$$y'' + \frac{3 \cdot p}{\left(p + x^2\right)^2} \cdot y = 0 \qquad y(a) = \psi(a) \qquad y(b) = \psi(b)$$

$Err = 0.1$

$sol := sb\_lbvp_2 \left(y, x, [a \;\; b], 1000 \right)$



Plot ("XY", sol)



Plot ("YY'", sol)

---

lbvp example

Example $[a \;\; b] := \left[\frac{1}{3 \cdot \pi} \;\; \frac{3}{\pi} \right]$

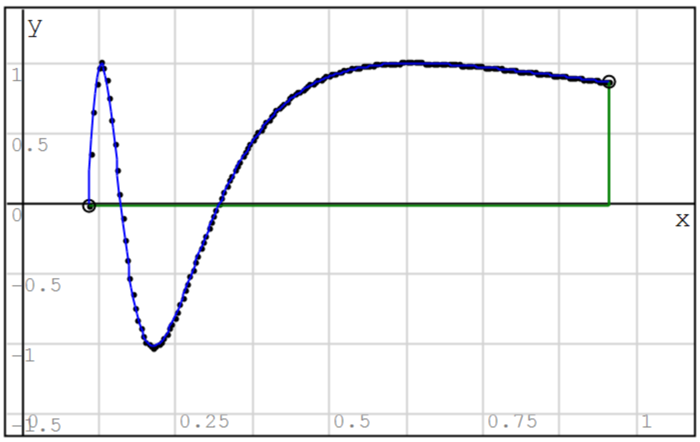
Numeric solution

$$y'' + \frac{2}{x} \cdot y' + \frac{y}{x^4} = 0 \qquad y(a) = 0 \qquad y(b) = \frac{\sqrt{3}}{2}$$

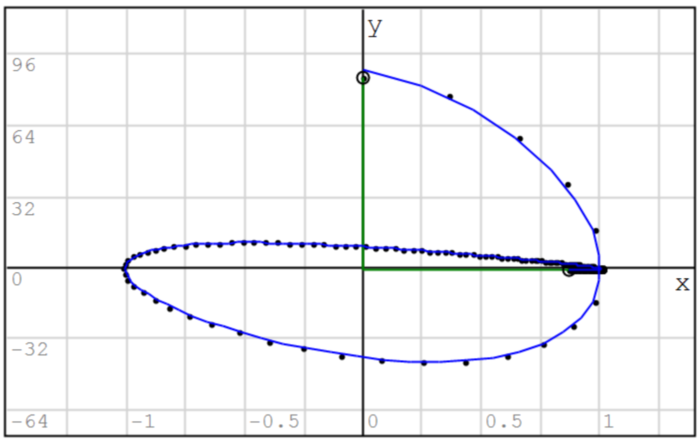
$sol := sb\_lbvp_2 \left(y, x, [a \;\; b], 200 \right)$

Analytic solution $\psi(x) := \sin\left(\frac{1}{x}\right)$

$Err = 0.11$



Plot ("XY", sol)



Plot ("YY'", sol)

---

bvp with two solutions

Example $[a \;\; b] := [0 \;\; 4]$ $\quad y_b := -2 \quad N := 50 \quad \varepsilon := 10^{-9}$

Numeric solution

$$y'' + |y| = 0 \qquad y(a) = 0 \qquad y(b) = y_b$$

$sol(Yo) := sb\_bvp_2 \left(y, x, [a \;\; b], N, Yo, \varepsilon \right)$

$sol := sol(0)$

$[X \;\; Y \;\; Y'] := Cols(sol)$

$\text{Clear}\left(y'_a\right) = 1$

This problem have two solutions. I compare the bvp solution with the shooting method, because can't found a symbolic expression.

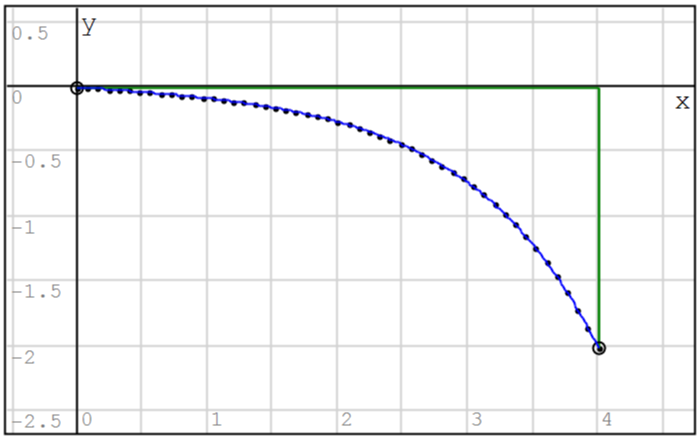
Shoot solution

$$\psi''(x) + |\psi(x)| = 0 \qquad \psi(a) = 0 \qquad \psi'(a) = y'_a$$

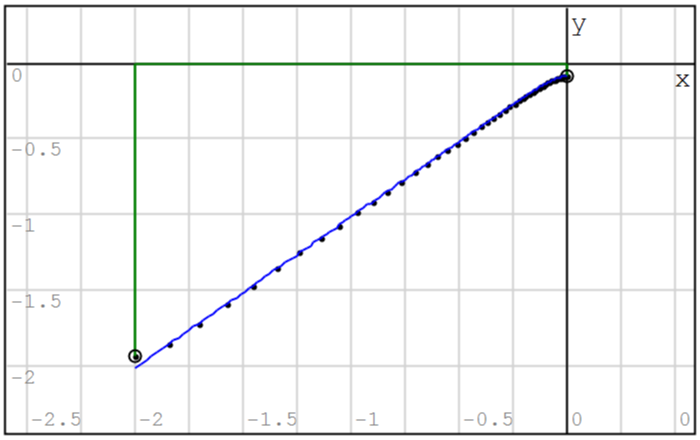
$shoot(y'_a) := \text{Rkadapt}(\psi(x), b, N)$

$Eq(y'_a) := shoot(y'_a)_{N+1\,2} - y_b \qquad y'_a := \text{al\_nleqsolve}(Y'_1, Eq)_1 = -0.0733$

$[\Xi\ \Psi\ \Psi'] := Cols(shoot(y'_a)) \qquad \psi'(x) := \text{cinterp}(\Xi, \Psi', x) \qquad Err = 0$



Plot("XY", sol)



Plot("YY'", sol)

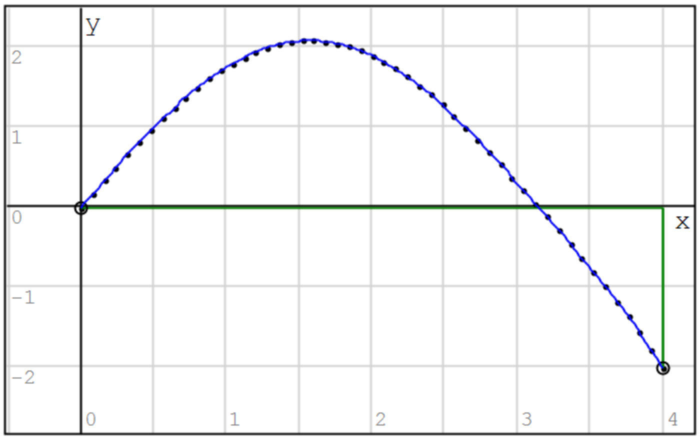
For the other solution: call bvp with a new guess

$sol := sol(1 + \text{matrix}(N+1, 1)) \qquad [X\ Y\ Y'] := Cols(sol)$

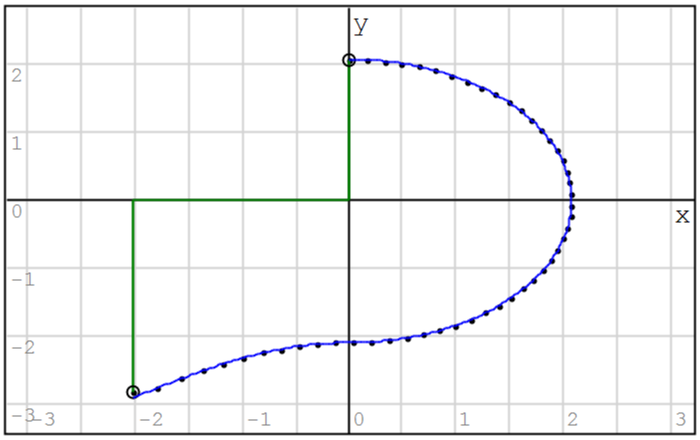
Solve the shoot with a new guess from bvp

$y'_a := \text{al\_nleqsolve}(Y'_1, Eq)_1 = 2.0666$

$[\Xi\ \Psi\ \Psi'] := Cols(shoot(y'_a)) \qquad \psi'(x) := \text{cinterp}(\Xi, \Psi', x) \qquad Err = 0.01$



Plot("XY", sol)



Plot("YY'", sol)

Shooting method

Example $[a\ b] := [-1\ 6] \qquad [y_a\ y_b] := [7\ 3] \qquad N := 100 \qquad \varepsilon := 10^{-9}$

Numeric solution

$$y'' + \frac{1}{2} \cdot y' + y = 5 \qquad y(a) = y_a \qquad y(b) = y_b$$

$[X\ Y\ Y'] := Cols(sol)$

$\text{Clear}(y'_a) = 1$

$sol := sb\_lbvp_2(y, x, [a\ b], N)$

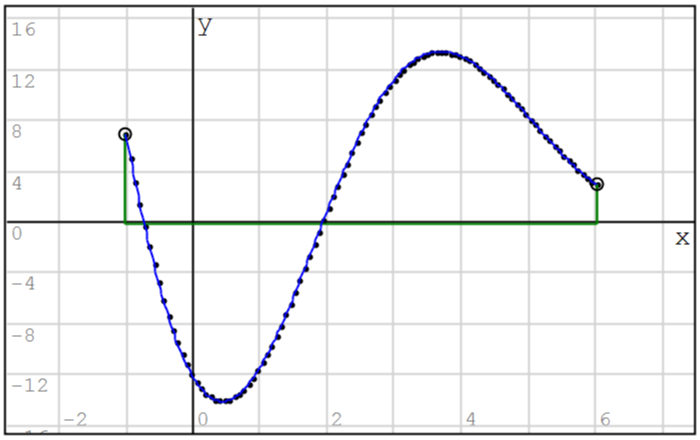
Shoot solution

$$\psi''(x) + \frac{1}{2} \cdot \psi'(x) + \psi(x) = 5 \qquad \psi(a) = y_a \qquad \psi'(a) = y'_a$$

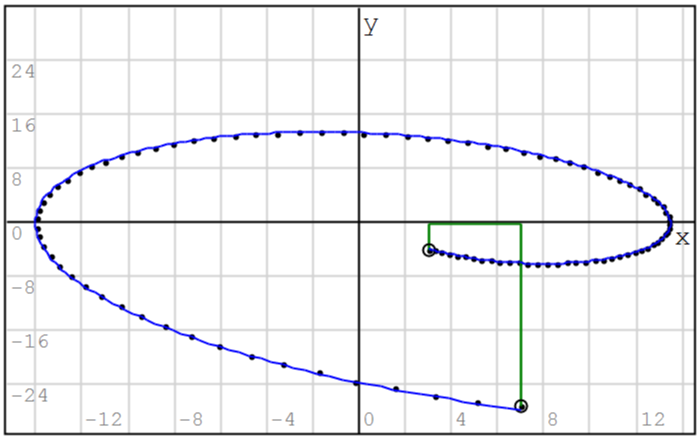
$$shoot(y'_a) := \text{Rkadapt}(\psi(x), b, N)$$

$$Eq(y'_a) := shoot(y'_a)_{N+1\,2} - y_b \qquad y'_a := \text{al\_nleqsolve}(Y'_1, Eq)_1 = -27.5704$$

$$[\,\Xi \;\; \Psi \;\; \Psi'\,] := Cols(shoot(y'_a)) \qquad \psi'(x) := \text{cinterp}(\Xi, \Psi', x) \qquad Err = 0.34$$



Plot("XY", sol)



Plot("YY'", sol)

Alvaro