Integration using ode solvers

Integrals can be evaluated with an appropiate ode solver. For example:

$\varphi(x) := x \cdot \cos(x)$ $\qquad$ $[a \ \ b] := [0 \ \ \pi]$

$$\int_a^b \varphi(x) \, dx = -2$$

```
D(x, y) := [φ(x)]                                  = -2.04951252
M := rkfixed([0], a, b, 100, D(x, y))
M
  rows(M) 2
```

Notice that Rkadapt fails, but not this other from alglib

```
D(x, y) := [φ(x)]                                  = -2
M := al_rkckadapt([0], a, b, 100, D(x, y))
M
  rows(M) 2
```

The ode solver

We can define our own integration procedure, cleaning the ode solver method for improve it's performance, This is a rude Runge Kuta 2-3 solver

```
RK23(u_0, t_1, t_2, N, E) :=
  [ u := u_0   t := t_1   h_o := eval((t_2 - t_1)/N)   h := h_o ]
  K(t, k) := | try
             |    eval(h·D(t))
             | on error
             |    eval(h·k)
  while t < t_2
     k_1 := K(t, u)
     k_2 := K(t + h, k_1)
     k_3 := K(t + 0.5·h, k_2)
     if (d := eval(|| (k_1 - 2·k_3 + k_2)/3 ||)) ≤ (q := eval(E·max([|u| 1])))
        t := t + h
        u := eval(u + (k_1 + 4·k_3 + k_2)/6)
     else
        ""
     if d ≠ 0
        h := eval(min([h_o  0.9·h·(q/d)^(1/3)]))
     else
        ""
  u
```

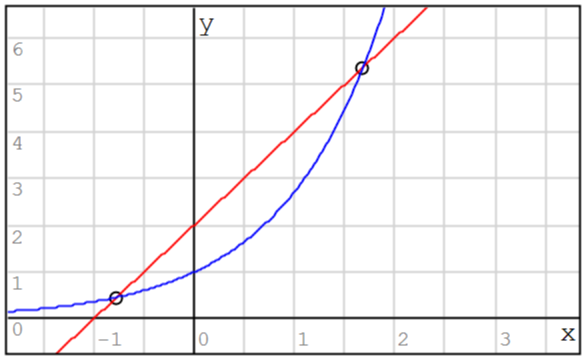
Thus

```
D(x) := φ(x)                          = -2.00113492
RK23(0, a, b, 100, 10^-6)
```

Double Integrals

SMath can Integrate $f(x, y) := x^2 + y^2$ over the region defined between $\begin{cases} a(x) := \mathbf{e}^x \\ b(x) := 2 \cdot x + 2 \end{cases}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \mathrm{sort}\left(\mathrm{solve}\left(b(x) - a(x), x\right)\right) = \begin{bmatrix} -0.768 \\ 1.6783 \end{bmatrix}$$



$$t_0 := \mathrm{time}(0)$$

$$I_0 := \mathrm{eval}\left(\int_{x_1}^{x_2} \int_{a(x)}^{b(x)} f(x, y) \, \mathrm{d}y \, \mathrm{d}x\right) = 18.57686864$$

$$\mathrm{time}(0) - t_0 = 1.891 \text{ s}$$

Define some utilities

$$Err\&T(t_0) := \left\| \begin{array}{l} \left[\, \Delta t := \mathrm{time}(0) - t_0 \quad t_0 := \mathrm{time}(0) \,\right] \\ \left[\dfrac{\left| I - I_0 \right|}{\Delta t} \right] \end{array} \right.$$

$\varepsilon_x := 10^{-3}$ $\varepsilon_y := 10^{-3}$

$n_x := 15$ $n_y := 15$

Now, we try to extend the above method for solve this problem parametrizing the ode solver from the SMath plugins ...

$$DInt\left(f(2), x_1, x_2, y_1, y_2, n_x, n_y\right) := \left| \begin{array}{l} RK(a, b, n) := \left| \begin{array}{l} M := \mathrm{al\_rkckadapt}\left([\,0\,], a, b, n, D(x, y)\right) \\ \mathrm{eval}\left(M_{\mathrm{rows}(M)\,2}\right) \end{array} \right. \\ D(y, u) := \left| \begin{array}{l} D(x, v) := \left[\, f(x, y) \,\right] \\ RK\left(x_1, x_2, n_x\right) \end{array} \right. \\ RK\left(y_1, y_2, n_y\right) \end{array} \right.$$

... or using our own ode solver ...

$$DInt\left(f(2), x_1, x_2, y_1, y_2, n_x, n_y, \varepsilon_x, \varepsilon_y\right) := \left| \begin{array}{l} D(y) := \left| \begin{array}{l} D(x) := f(x, y) \\ RK23\left(0, x_1, x_2, n_x, \varepsilon_x\right) \end{array} \right. \\ RK23\left(0, y_1, y_2, n_y, \varepsilon_y\right) \end{array} \right.$$

... and, as first approach, using integration over a rectangle bigger than the true region of integration and making zero f outside the region of integration

$g(x, y) := f(x, y) \cdot (y \le b(x)) \cdot (y \ge a(x))$ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} := \begin{bmatrix} 0 \\ 6 \end{bmatrix}$ $t_0 := \mathrm{time}(0)$

$x_2 \; y_2$

$Err\&T(t_0) = \begin{bmatrix} 0.0064 \\ 5.35 \text{ s} \end{bmatrix}$

$$I := \int_{x_1}\int_{y_1} g(x, y)\,dy\,dx = 18.5832591$$

$$I := DInt\left(g(x, y), x_1, x_2, y_1, y_2, n_x, n_y\right) = \blacksquare$$

Convergent only for rkafixed ode solver.

$$I := DInt\left(g(x, y), x_1, x_2, y_1, y_2, n_x, n_y, \varepsilon_x, \varepsilon_y\right) = 18.45199634 \qquad Err\&T\left(t_0\right) = \begin{bmatrix} 0.1249 \\ 71.53\ s \end{bmatrix}$$

As better approach, using a change of variables $y = y_1 + u \cdot \left(y_2 - y_1\right)$ $dy = \left(y_2 - y_1\right) \cdot du$

we can integrate from 0 to 1 for y, and isn't necessary to guess the limits of integration, as in the first method.

$$h(x, u) := f\left(x, a(x) + u \cdot (b(x) - a(x))\right) \cdot (b(x) - a(x))$$

$$I := \int_{x_1}^{x_2}\int_{0}^{1} h(x, y)\,dy\,dx = 18.56839179 \qquad Err\&T\left(t_0\right) = \begin{bmatrix} 0.0085 \\ 10.453\ s \end{bmatrix}$$

$$I := DInt\left(h(x, y), x_1, x_2, 0, 1, n_x, n_y\right) = 18.57686894 \qquad Err\&T\left(t_0\right) = \begin{bmatrix} 3.0316 \cdot 10^{-7} \\ 7.387\ s \end{bmatrix}$$

$$I := DInt\left(h(x, y), x_1, x_2, 0, 1, n_x, n_y, \varepsilon_x, \varepsilon_y\right) = 18.4681218 \qquad Err\&T\left(t_0\right) = \begin{bmatrix} 0.1087 \\ 2.139\ s \end{bmatrix}$$

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