PAPER

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SMath studio modelling of alpha-particle scattering

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Abstract

In this paper it is demonstrated how the free, and easily downloadable, software package called SMath Studio can be used to set up a model of alpha-particle scattering. The basic physics of the motion of an alpha-particle in the nuclear coulomb field is used to produce a simple stepwise computer algorithm which, in conjunction with a novel set of units, is incorporated into a SMath worksheet with graphical output. It is also shown how the computed results can be compared with traditional theory.

Keywords: alpha particle, SMath, Rutherford, scattering

1. Introduction

In 1909, Hans Geiger and Ernest Marsden found that alpha particles occasionally scatter at large angles when passing through a thin layer of material (e.g. gold film). This and other experiments (see the interesting paper by Leone, Robotti and Verna [1]) led Ernest Rutherford in 1912 to formulate his theory of the nuclear atom. The topic is widely described in physics textbooks, where the treatment varies from the qualitative to the densely mathematical analyses found in more advanced courses.

Before detailed accounts of scattering experiments can be understood, students need a quantitative appreciation of the motion of a single alpha particle in a central coulomb field. Obtaining meaningful results traditionally requires non-trivial mathematical analysis to determine the scattering angle dependency on the impact parameter ('aiming error'). The treatment below shows how, by making use of very basic physics and mathematics, and a step-wise computer approach, it is possible for students to come to grips with this important topic.

2. The system equations

Figure 1 shows an alpha particle of charge Q and mass m in the electric field of a nucleus of charge Q_N. The alpha particle has coordinates (x, y) and is at a distance r from the nucleus. F is the repulsive force on the alpha particle.

The force F is given by Coulomb's law and Newton's second law:

$$F = \frac{Q_N Q}{4\pi\varepsilon_0 r^2} \text{ and } F = m\frac{dv}{dt}$$

where v is the velocity of the alpha particle.

It immediately follows that the alpha's acceleration is given by

$$\frac{dv}{dt} = \frac{k}{r^2} \text{ where } k \text{ is a constant}$$

$$\text{i.e. } k \equiv \frac{Q_N Q}{4\pi\varepsilon_0 m} \tag{1}$$

The x- and y-components of the acceleration are then

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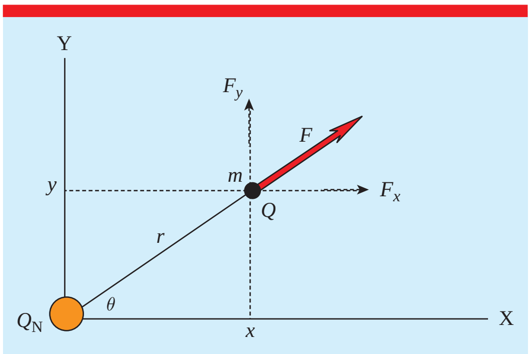

Figure 1. Alpha particle in field of nucleus.

$$\frac{dv_x}{dt} = \frac{kx}{r^3}$$

$$\frac{dv_y}{dt} = \frac{ky}{r^3}$$

where $$r = \sqrt{(x^2 + y^2)}$$

and $$\frac{dx}{dt} = v_x$$

$$\frac{dy}{dt} = v_y$$

Figure 2. The system equations.

$$\frac{dv_x}{dt} = \frac{k}{r^2}\cos\theta = \frac{k}{r^2}\left(\frac{x}{r}\right) = \frac{kx}{r^3}$$

$$\frac{dv_y}{dt} = \frac{k}{r^2}\sin\theta = \frac{k}{r^2}\left(\frac{y}{r}\right) = \frac{ky}{r^3}.$$

We thus have a set of equations (figure 2) which completely specify the system:

3. Solving the system equations

The equations in figure 2 can be solved using the methods of integral calculus or by applying well known sophisticated methods of numerical integration.

However, a simple step-wise approach can provide sufficient precision for our purposes. Taking small ***finite*** time steps Δt, we can write the approximations:

$$\Delta v_x \approx \frac{kx}{r^3}\,\Delta t,\ \Delta v_y \approx \frac{ky}{r^3}\,\Delta t, \text{ etc.}$$

This gives a set of expressions providing a means of updating x and y as a function of time t.

$$x_{new} \leftarrow x_{old} + v_{x,old}\,\Delta t$$

$$y_{new} \leftarrow y_{old} + v_{y,old}\,\Delta t$$

$$r \leftarrow \sqrt{x_{old}{}^2 + y_{old}{}^2}$$

$$v_{x,new} \leftarrow v_{x,old} + k\frac{x_{new}}{r^3}\,\Delta t$$

$$v_{y,new} \leftarrow v_{y,old} + k\frac{y_{new}}{r^3}\,\Delta t$$

$$t_{new} \leftarrow t_{old} + \Delta t.$$

The whole process has to begin, of course, with initial values of t, x, y, v_x, and v_y.

4. The question of units

Students should, of course, be familiar with International System of Units (SI) units, but need to be aware of the ***scale*** of the physical situation under consideration, and use units appropriate to that scale.

It is perfectly possible, for example to express distances on all scales purely in terms of the metre.

For example, the Earth–Sun distance is 1.49×10^{11} m and the radius of the bromine atom is 1.14×10^{-10} m.

However, it is far neater to state these quantities as 149 Gm and 114 pm respectively, but for calculations, one normally has to revert to standard-form to ensure that computed values come out correctly. This is because a ***coherent*** system is necessary, and SI is one such system.

In the past, physicists have, on occasion chosen units which are easier to handle for a particular scale. e.g. Hartree units, where the electron mass and charge are taken to be unity. However this type of system does not have a simple relation to SI. Nonetheless, other coherent and SI-related systems are possible. I have discussed this idea in a previous paper [2].

5. SN units

For our present needs, it has proved convenient to use what I have called SN units. This system is ***coherent*** and is very easy to use at the nuclear scale. The base units are as follows:

Mass yoctogram (yg) = 10^{-24} g = 10^{-27} kg
Length femtometre (fm) = 10^{-15} m
Time zeptosecond (zs) = 10^{-21} s
Charge attocoulomb (aC) = 10^{-18} C

The unit of energy is derived as the femtojoule (fJ).

It is left as an exercise for the interested reader to show that in this system the constant *k,* for gold, on page 2 has the value 5476 fm^3 zs^{-2}.

6. Closest approach

Elastic scattering of the alpha particle means that energy is conserved, i.e. the kinetic energy of the alpha ‘at infinity’ is equal to its potential energy at the distance of closest approach r_{min}.

So, we can write:

$$\frac{1}{2}m{v_0}^2 = \frac{Q_N Q}{4\pi\varepsilon_0 r_{min}}$$

where v_0 is the speed of the alpha ‘at infinity’.

Now, substituting the definition of k on page 2, we obtain

$$r_{min} = \frac{2k}{{v_0}^2} \tag{2}$$

We shall use this in our model.

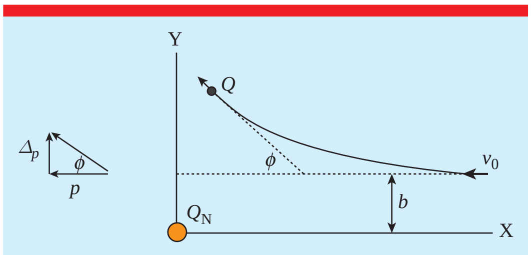

Figure 3. Scattering angle.

7. The scattering angle φ

With reference to figure 3, students can be shown an ‘arm-waving’ estimate of the scattering angle φ.

The alpha particle ‘at infinity’ has mass *m,* velocity v_0 and impact parameter b. Its momentum $p = mv_0$.

The time it spends in the vicinity of the nucleus must be on the order τ.
where $\tau \sim \frac{b}{v_0}$ as $\frac{b}{v_0}$ has the dimensions of time.

(Typically if $b = 10$ fm and $v_0 = 10$ fm zs^{-1}, then $\tau = 1\ zs$)

Also during this time, the repulsive force normal to the original trajectory is

$$F \sim \frac{Q_N Q}{4\pi\varepsilon_0 b^2}.$$

So, the transverse momentum imparted to the alpha particle will be

$$\Delta p \sim F.\,\tau = \frac{Q_N Q}{4\pi\varepsilon_0 b^2}\left(\frac{b}{v_0}\right) = \frac{Q_N Q}{4\pi\varepsilon_0 b v_0}.$$

The scattering angle $\varphi = \frac{\Delta p}{p} \sim \frac{Q_N Q}{4\pi\varepsilon_0 b v_0}\cdot\frac{1}{mv_0}$

So finally, we expect $\varphi \sim \frac{Q_N Q}{4\pi\varepsilon_0 m{v_0}^2 b}$

A rigorous treatment by French [3] yields the expression

$$\tan\frac{\varphi}{2} = \frac{Q_N Q}{4\pi\varepsilon_0 m{v_0}^2 b}. \tag{3}$$

Combining equations (1)–(3), we find the scattering angle to be

$$\varphi = 2.\arctan\left(\frac{r_{min}}{2b}\right) \tag{4}$$

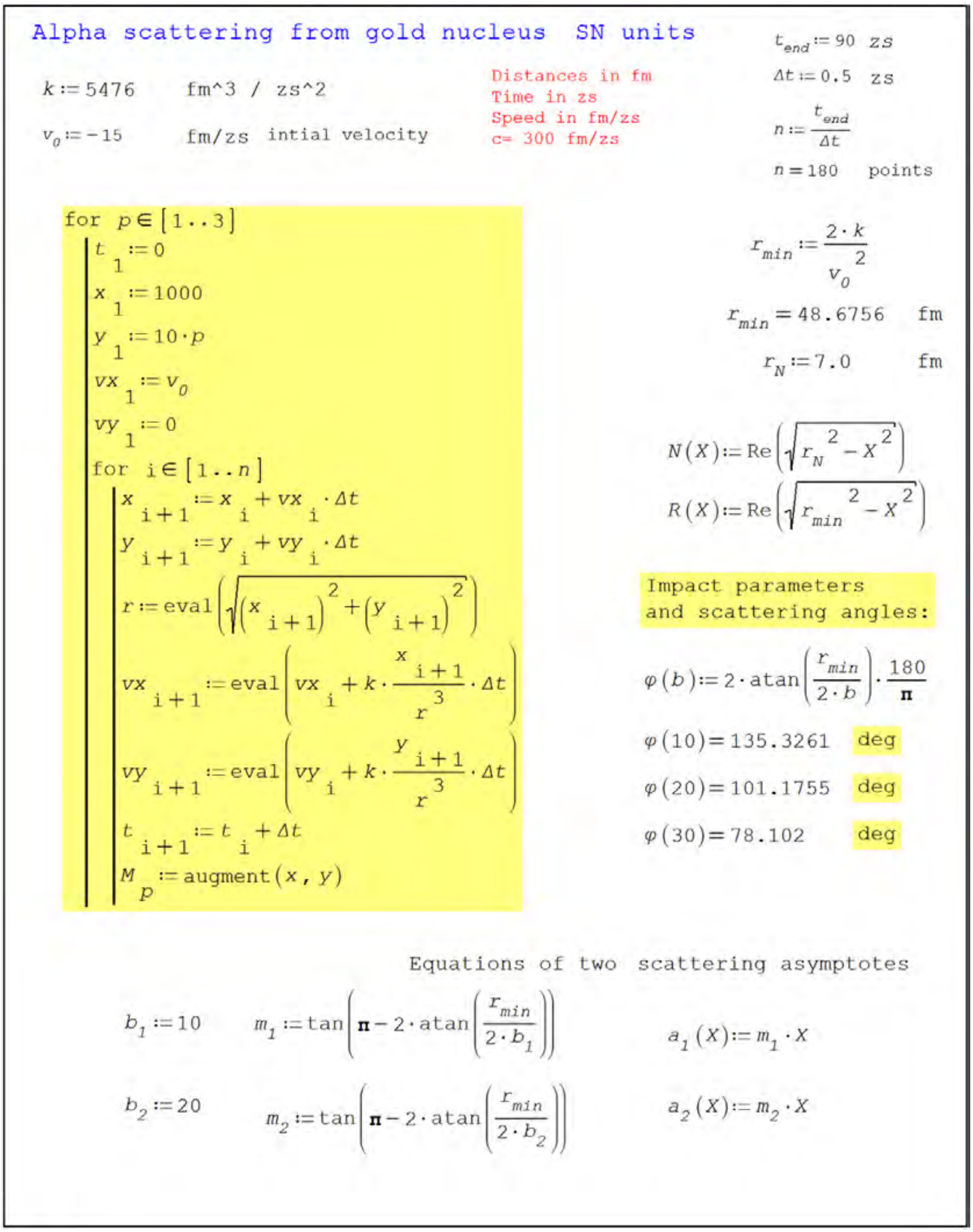

Figure 4. (a) SMath worksheet.

8. The SMath model

Figure 4(a) shows the first half of the SMath worksheet the main part of which is the inner ***for***-loop which updates the values of x, y, r, v_x, v_y, and t.

The number of plotting points n is found from the total time t_{end} and the time step Δt. The subscript variable i runs from 1 to n. The *eval* function is a peculiarity of SMath which simply serves to speed up the computation.

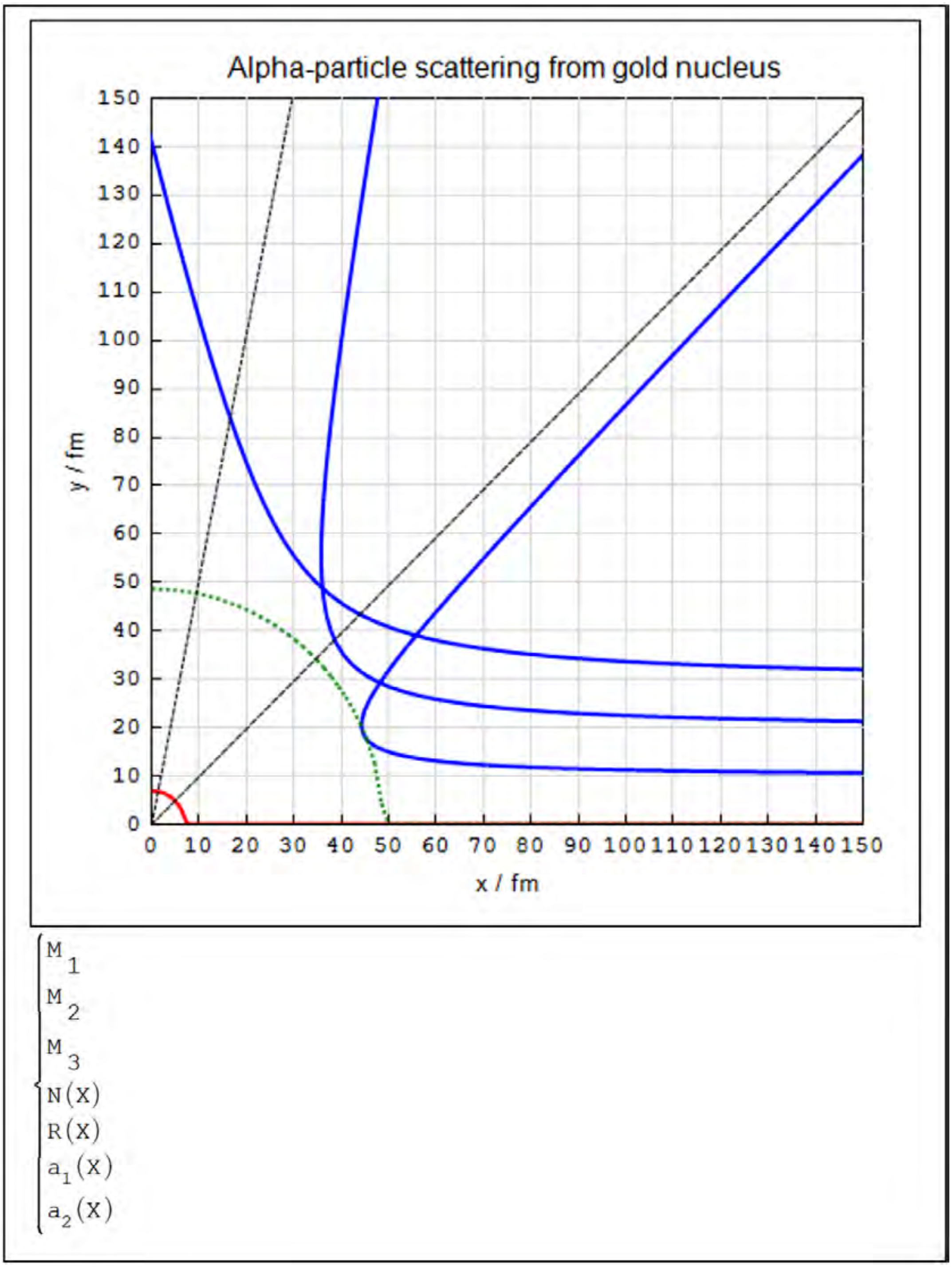

Figure 4. (b) SMath worksheet graph section

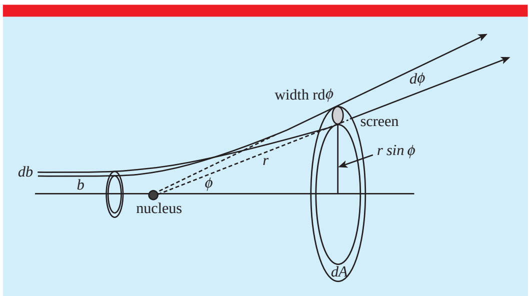

Figure 5. Scattering geometry.

The outer ***for***-loop is controlled by the loop variable p which runs over the range 1 to 3, generating three different impact parameters 10 fm, 20 fm, and 30 fm.

The last line in the inner loop generates three matrices M1, M_2, and M_3 which contain the x and y values for plotting the trajectories.

The functions N(X) and R(X) generate part-circles to represent the nucleus and the radius of closest approach, respectively, X being used here to avoid confusion with x in the main algorithm. The nuclear radius for gold is known to be 7.0 fm, and the model calculated the distance of closest approach to be 50 fm. 'Re' takes the real part of a function.

Typical alpha particle energies in this context are 800 fJ. This implies

a speed 'at infinity' of about 15 fm zs^{-1}. This is used in our model.

Figure 4(b) shows the resulting trajectories.

Equation (4) was used to calculate scattering angle φ for the three chosen impact parameters.

It was also decided to compute the scattering asymptotes for the first two impact parameters (10 fm and 20 fm).

The gradient m of an asymptote is clearly tan(π—φ) which, using equation (4), gives

$$m = \tan\left\{\pi - 2.\arctan\left(\frac{r_{min}}{2b}\right)\right\}.$$

The asymptote equation is

$$a(X) \ = m.X.$$

Right clicking on the SMath X-Y plot area opens a menu which allows formatting of the axes, plotting colours and types of line.

The trajectories are in blue, the nucleus in red, closest approach radius in green and the asymptotes in black.

It is important to consider an appropriate value for x_1—the starting distance of the alpha particle. In theory this is infinite, but students and teachers can profitably experiment with this. The value of 1000 fm was found to give sensible results. If x_1 is too big, the computation time becomes very large—too small and the trajectories will be plotted inside $r_{\min}$! A run time of around 100 zs was found to work well. Consideration should also be given to the size of the time step Δt. In figure 4(a), a timestep of 0.5 zs with a run time of 90 zs was used, giving 180 computed points.

If the graph in figure 4(b) is printed out (making sure that x- and y-axes have the same scale), students can measure the angles made by the asymptotes to verify the predicted scattering angles.

Details on how to use SMath are to be found in [4, 5].

9. Scattering experiments

The above analysis forms a solid basis from which students may proceed to examine the alpha-particle scattering experiments carried out by Geiger and Marsden, the results of which strongly supported Rutherford's theory of the nuclear atom.

Equation (3) can be rearranged to yield the expression below:

$$b = \frac{k}{{v_0}^2}\, cot\frac{\varphi}{2}. \tag{5}$$

Considering a range of impact parameters (figure 5) from b to $b + db$ with corresponding scattering angles from φ to $\varphi + d\varphi$, we can imagine a number dN of particles scattered through this region via a ring of circumference $2\pi b$ and width db.

Differentiating (5) gives

$$db = \frac{k}{2{v_0}^2}\, csc^2\frac{\varphi}{2}\, d\varphi. \tag{6}$$

Also,

$$dN = I_0.\, 2\pi b.db. \tag{7}$$

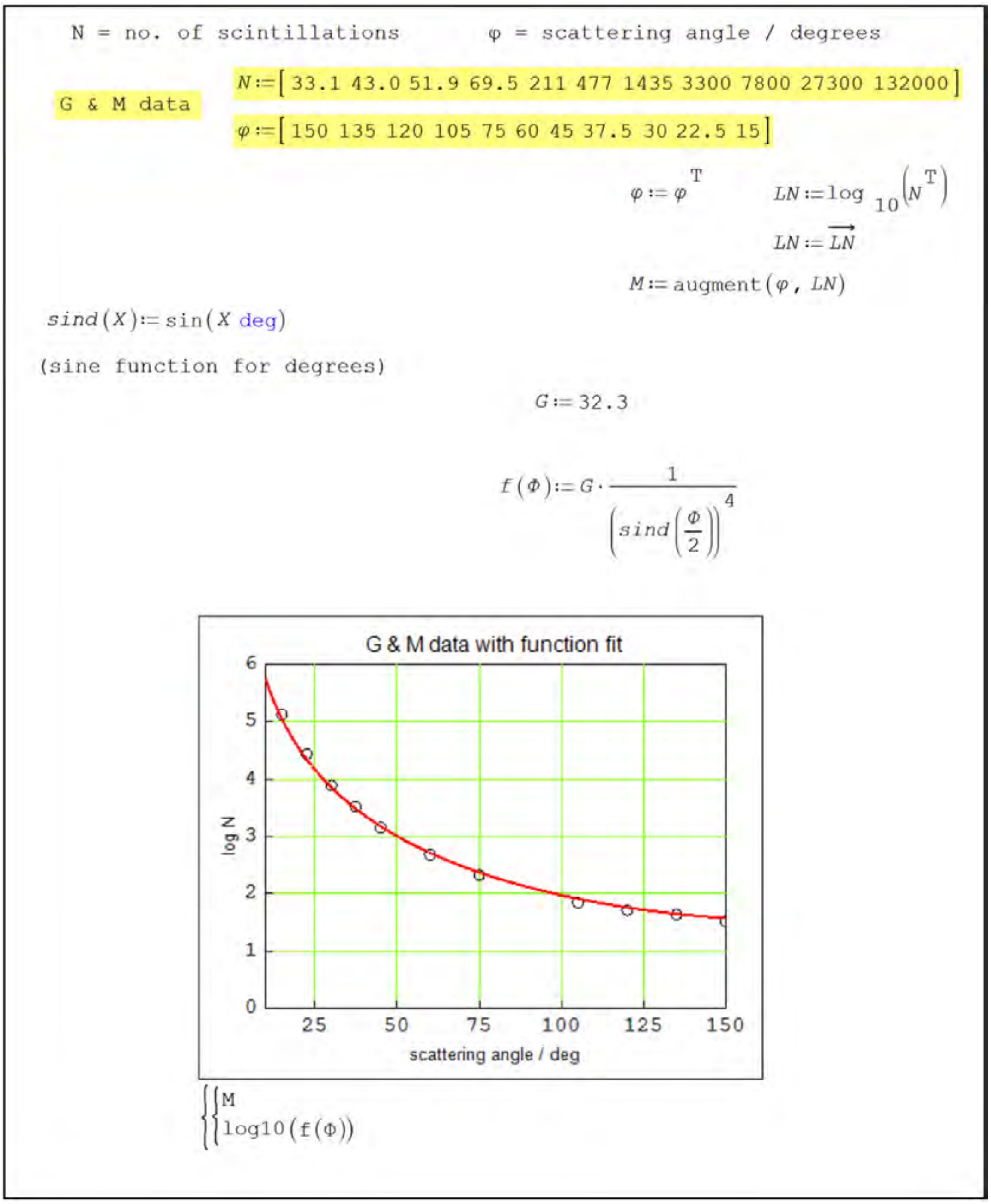

Figure 6. SMath sheet for G&M data test.

where I_0 is the incident intensity of the alpha-particle beam.

Substitution of (5) and (6) into (7) results in the expression

$$dN = \frac{I_0 \pi k^2}{v_0{}^4} \frac{cos\frac{\varphi}{2}}{sin^3\frac{\varphi}{2}} d\varphi. \tag{8}$$

These dN particles pass through an area dA which is part of a spherical surface of radius $r\sin\varphi$ and width $rd\varphi$ so that

$$dA = 2\pi.rsin\varphi.rd\varphi. \tag{9}$$

Combining (8) and (9), after a little algebra, we find

$$\frac{dN}{dA} = \frac{I_0 k^2}{4v_0{}^4 r^2 sin^4\frac{\varphi}{2}}$$

which finally leads to the expression

$$N = \frac{nt.A_s I_0 k^2}{4v_0{}^4 r^2 sin^4\frac{\varphi}{2}} \tag{10}$$

where N is the number of particles, in a standard interval, detected on a scintillation screen of area A_s. The quantities n and t are the number density of atoms in the thin target of thickness t.

The $1/sin^4\frac{\varphi}{2}$ dependency in equation (10) was verified by Geiger and Marsden in 1913.

A useful student exercise is the comparison of scattering data from Geiger and Marsden's 1913 paper with the predicted $1/sin^4\frac{\varphi}{2}$ dependency in equation (10). A SMath worksheet illustrating this exercise is shown in figure 6. The data for N and φ are for a gold target and have been taken from the table on p 611 in Geiger and Marsden's paper [6]. The scattering angles are quoted in degrees and so the function 'sind' is needed because the SMath sine function assumes angles in radians. As the range of N values is large, it is necessary to plot log N rather than N. It would be left as a student task to find the necessary value of the constant G.

It can be seen that a very pleasing fit between data and theory is obtained.

10. Conclusion

I have demonstrated how the free software package SMath Studio can be used to model the classical elastic scattering of alpha particles. The models can be easily modified and used either in student investigations or as live simulations in a lecture-demonstration environment. One of the advantages of this approach is that the models are transparent, the equations producing the numerical solutions being clearly visible at all times.

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