⊞— Sets

⊟— Exercise

Conditional Probability

This exercise shows that probabilities are a measuring theory, and the importance of make experiments to have the actual values for physical or engieneering events, instead of the simple assignation of theoretical probability values for those events.

The universe of events for a die is

$D := [\text{"One" "Two" "Three" "Four" "Five" "Six"}]$

indexed by

$ID := [1..L(D)]^T$ with empty set $\varnothing := \|[\,0\,]$

Thus, the row vector

$C := [1\ 3\ 3]$

represents the event for get One, then Three and then Three again

$D_{1\,C} = [\text{"One" "Three" "Three"}]$

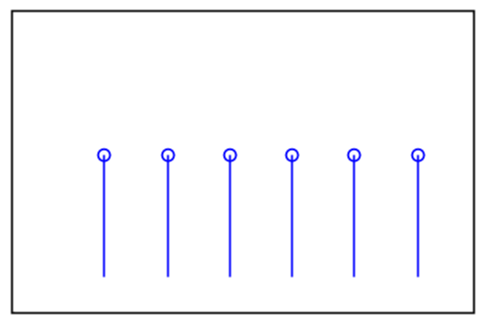
The probability for an event given by a row vector X is

Clear(P)=1

$P_D(X) :=$ try
  $product(P_X)$
on error
  0

where P is the probability of each die number. For a perfect die we have

$P := \left[\frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\right]$



PlotStem(ID, *P*)

then the probability for C is

$P_D(C) = \frac{1}{216}$

this is

$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$

Now, if rolling the same die twice, we have

$D2 := S(D, \times, D) = [[\text{"One" "One"}]\ [\text{"One" "Two"}]\ \cdots]$

and the indices are

$ID2 := [1..L(D2)]^T = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ \cdots]$

For correlate both universes define φ as

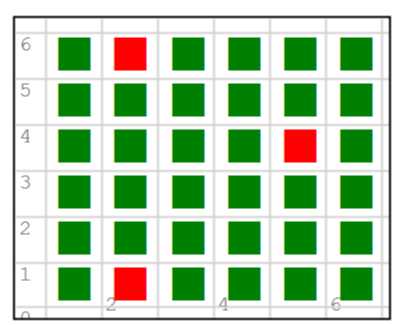
$ID2 := S(ID, \times, ID)$ $\varphi(X) := ID2_{1\,X}$

For example, the row vector C

$C := [7\ 28\ 12]$

maped by

$\varphi(C) = [[2\ 1]\ [5\ 4]\ [2\ 6]]$



Plot2Die(*C*)

represents the event of get Two and One, or Five and Four, or Two and Six

$D2_{1\,C} = [[\text{"Two" "One"}]\ [\text{"Five" "Four"}]\ [\text{"Two" "Six"}]]$

The probability in the 'rolling die twice' space is

$$P_{2S} := S(P, \times, P)$$

$$P_{2S}(X) := \begin{array}{l} \text{try} \\ \quad \displaystyle\sum_{\kappa=1}^{L(X)} product\left(P_{2S}, X_{\kappa}\right) \\ \text{on error} \\ \quad 0 \end{array}$$

It's well defined because in the universe 'ID2 because for each $X \subseteq ID2$ is

1 $P_{2S}(ID2) = 1$

2 $P_{2S}(\emptyset) = 0$

3 $0 \le P_{2S}(X) \le 1$

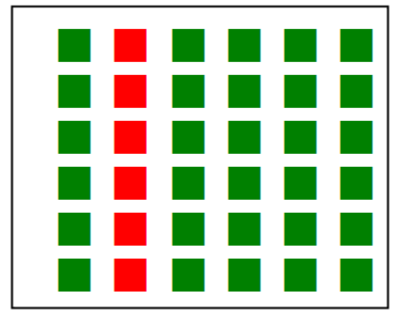
For the event given by C, it is $P_{2S}(C) = \frac{1}{12}$ or $\frac{3}{36} = \frac{1}{12}$

Defining an event A by 'first roll is two', we have

$$f_A(X) := \varphi(X)_1 = 2 \qquad A := Find_1\left(\overrightarrow{f_A(ID2)}\right)^T = [\,7\ 8\ 9\ 10\ 11\ 12\,]$$

$$P_{2S}(A) = \frac{1}{6} \qquad \frac{6}{36} = \frac{1}{6}$$

$D2_{1\,A}$ = [["Two" "One"] ["Two" "Two"] ["Two" "Three"] ⋯]



Plot2Die(A)

Defining an event B by 'the sum for two rolls is less or equal than five', we have

$$f_B(X) := \varphi(X)_1 + \varphi(X)_2 \le 5 \qquad B := Find_1\left(\overrightarrow{f_B(ID2)}\right)^T = [\,1\ 2\ 3\ 4\ 7\ 8\ \cdots]$$

$$P_{2S}(B) = \frac{5}{18} \qquad \frac{10}{36} = \frac{5}{18}$$

$D2_{1\,B}$ = [["One" "One"] ["One" "Two"] ["One" "Three"] ⋯]

Plot2Die(B)

The event given by the intersection of the two previous cases is

$$P_{2S}(S(A, \cap, B)) = \frac{1}{12} \qquad \frac{3}{36} = \frac{1}{12}$$

$D2_{1\,S(A,\cap,B)}$ = [["Two" "One"] ["Two" "Two"] ["Two" "Three"]]

Plot2Die(S(A, ∩, B))

Finally, the conditional probability P(B|A) is

$$P_{B|A} := \frac{P_{2S}(S(A, \cap, B))}{P_{2S}(B)} = \frac{3}{10}$$

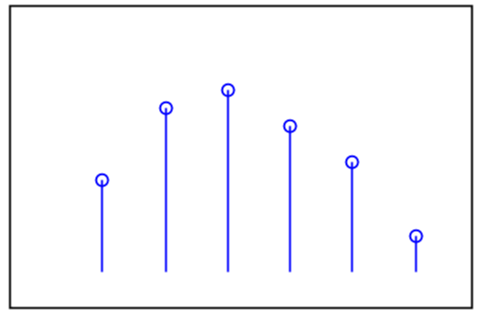
That's ok for a perfect die. What about a real one? It's easy to make one, with a small attached mass, for example, in the six face. Rolling it 40 times, I found this events

$E\$:=$

"Two"	"Three"	"Three"	"Five"	"Four"	"Three"	"Five"	"One"	"Two"	"Three"
"Five"	"Two"	"Five"	"One"	"Three"	"Five"	"Four"	"Six"	"Two"	"Three"
"Four"	"Two"	"One"	"Four"	"Five"	"One"	"Two"	"Four"	"Three"	"Four"
"Three"	"Two"	"One"	"Two"	"Four"	"Three"	"Four"	"Six"	"Three"	"Two"

The probability for each number is now $N := L(E\$) = 40$

$$P_{1\,ID} := \frac{L\left(\text{findrows}\left(E\$_{[1..N]}, D_{ID}, 1\right)\right)}{N}$$



PlotStem(ID, P)

$$P = [\,0.125\ 0.225\ 0.25\ 0.2\ 0.15\ 0.05\,]$$ with $$\sum P = 1$$

For this imperfect die now we have:

$P_{2S}(A) = 0.225$ instead $\frac{6}{36} = 0.1667$

$P_{2S}(B) = 0.3475$ $\frac{10}{36} = 0.2778$

$P_{2S}(S(A, \cap, B)) = 0.135$ $\frac{3}{36} = 0.0833$

$$P_{B|A} := \frac{P_{2S}(S(A, \cap, B))}{P_{2S}(B)} = 0.3885$$ $\frac{3}{10} = 0.3$

Exercise Here, we roll the same die twice. Make a second imperfect die, and eval the conditional probability for the same events, but rolling each die once.

Rolling my second imperfect die, with a small mass at the face of Five, 30 times, I have

$$E\$_2 := \begin{bmatrix} \text{"Six"} & \text{"One"} & \text{"Two"} & \text{"Two"} & \text{"Three"} & \text{"Six"} & \text{"Two"} & \text{"Five"} & \text{"One"} & \text{"Four"} \\ \text{"Four"} & \text{"Two"} & \text{"Four"} & \text{"Three"} & \text{"Four"} & \text{"One"} & \text{"Two"} & \text{"Five"} & \text{"Two"} & \text{"Six"} \\ \text{"Three"} & \text{"One"} & \text{"Two"} & \text{"Four"} & \text{"Two"} & \text{"Two"} & \text{"Six"} & \text{"Four"} & \text{"Six"} & \text{"Two"} \end{bmatrix}$$

Demostrante using SMath that now it is $P_{B|A} = \frac{153}{440}$

References https://en.wikipedia.org/wiki/Conditional_probability#Example

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