⊞—Utilities

⊞—bvp.2

Boundary Values Problem

$lbvp_2$ $\quad lbvp_2(f(x), x, M, N)$ solves the linear ODE $y'' + p \cdot y' + q \cdot y = r$ in $x = [a \; b]$

subject to the boundary conditions

$$\begin{cases} \alpha_1 \cdot y(a) + \beta_1 \cdot y'(a) = c_1 \\ \alpha_2 \cdot y(b) + \beta_2 \cdot y'(b) = c_2 \end{cases}$$

f is such that $f(x) = [p(x) \; q(x) \; r(x)]$ and $M = \begin{bmatrix} \alpha 1 & \beta 1 & c1 \\ \alpha 2 & \beta 2 & c2 \end{bmatrix}$

bvp_2 $\quad bvp_2(\varphi(x, y, y'), x, 0, M, N, \varepsilon)$ solves the non linear ODE $y'' = \varphi(x, y, y')$ in $x = [a \; b]$

subject to the the same boundary conditions, with Yo as guess for the solution with dimension N+1..
If Yo≡0, bvp.2 try with a line between f(a) and f(b) as guess.. ε is used as the tolerance for a Newton solver.

⊞—Short hands for plots

⊟—lbvp example

Example Solve $y'' + 2 \cdot h \cdot y' + (h^2 + h') \cdot y = 0$ in $[a \; b] := [0 \; 5]$ htt

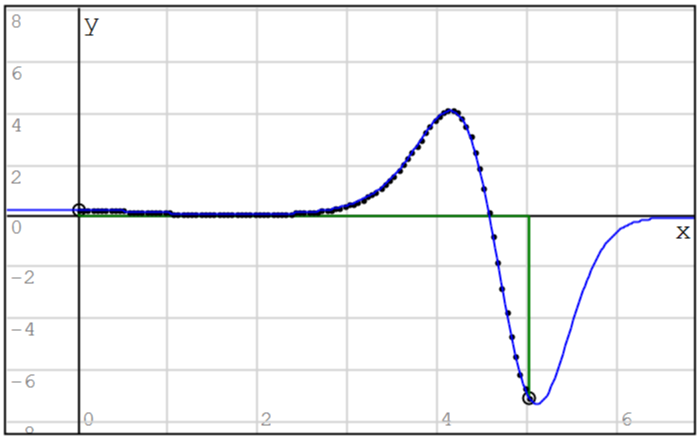
subject to the boundary conditions $bc := \begin{cases} y(a) + 3 \cdot y'(a) = 0.1 \\ y'(b) = -6 \end{cases}$

Numeric solution $\quad h := x \cdot \cos(x) \quad h' := \frac{d}{dx} h \quad f(x) := [2 \cdot h \;\; h^2 + h' \;\; 0] \quad M := \begin{bmatrix} 1 & 3 & 0.1 \\ 0 & 1 & -6 \end{bmatrix}$

Analytic solution $\quad y(x) := (A + B \cdot x) \cdot e^{-(x \cdot \sin(x) + \cos(x))} \quad y'(x) := \frac{d}{dx} y(x) \quad [N \; \varepsilon] := [50 \;\; 10^{-3}]$

$$\begin{bmatrix} A \\ B \end{bmatrix} := \text{roots}\left(\text{sys2mat}_1(bc), \begin{bmatrix} A \\ B \end{bmatrix}\right)$$

Results $\quad [X \; Y \; Y'] := Cols(lbvp_2(f(x), [a \; b], M, 100)) \quad Err = 0.16$



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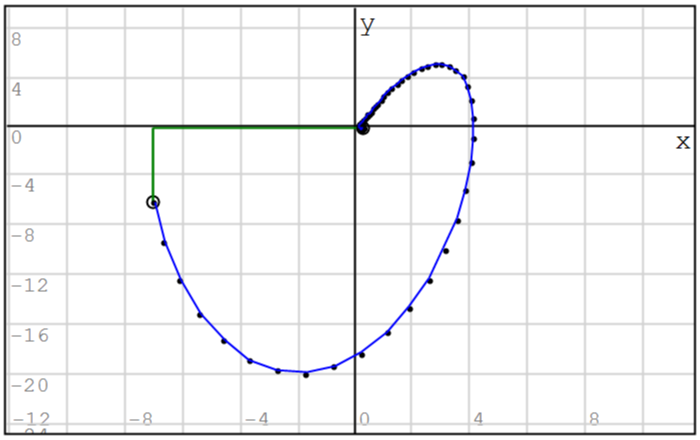


PlotYY'

Note Another way to solve for the integration constants is $\quad \text{Clear}(A, B) = 1$

$$eqM := \begin{bmatrix} M_{11} \cdot y(a) + M_{12} \cdot y'(a) - M_{13} \\ M_{21} \cdot y(b) + M_{22} \cdot y'(b) - M_{23} \end{bmatrix}$$

$$\text{roots}\left(eqM, \begin{bmatrix} A \\ B \end{bmatrix}\right) = \begin{bmatrix} 0.7939 \\ -0.174 \end{bmatrix}$$

$\text{Clear}(y(x), y'(x), A, B) = 1$

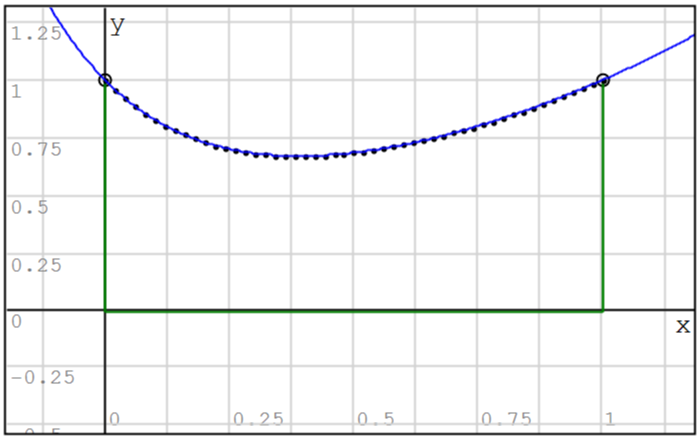
⊟—lbvp example

Example Solve $y'' + 3 \cdot y' - 4 \cdot y = 0$ in $[a \ b] := [0 \ 1]$ subject to the boundary conditions $\begin{cases} y(0) = 1 \\ y(1) = 1 \end{cases}$

Numeric solution $f(x) := [3 \ -4 \ 0]$ $M := \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Analytic solution
$$y(x) := \frac{\left(e^{(4 - 4 \cdot x)} + e^{x} + e^{(x+1)} + e^{(x+2)} + e^{(x+3)}\right)}{\left(1 + e + e^2 + e^3 + e^4\right)} \qquad y'(x) := \frac{d}{dx} y(x)$$

Results $[X \ Y \ Y'] := Cols\left(lbvp_2\left(f(x), [a \ b], M, 50\right)\right)$ $Err = 0$



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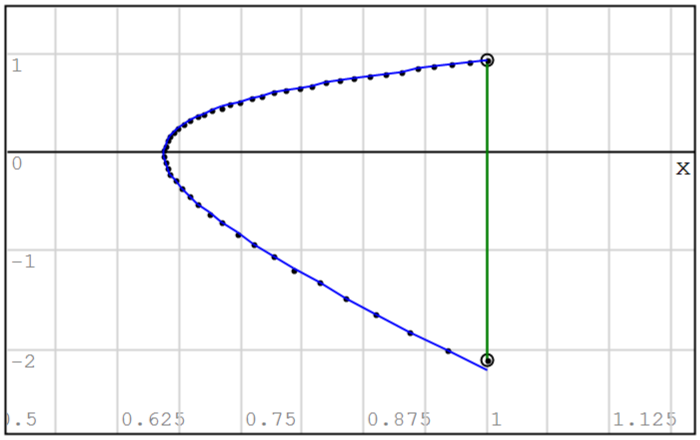


PlotYY'

$\text{Clear}(x, y(x), y'(x)) = 1$

⊟—lbvp example

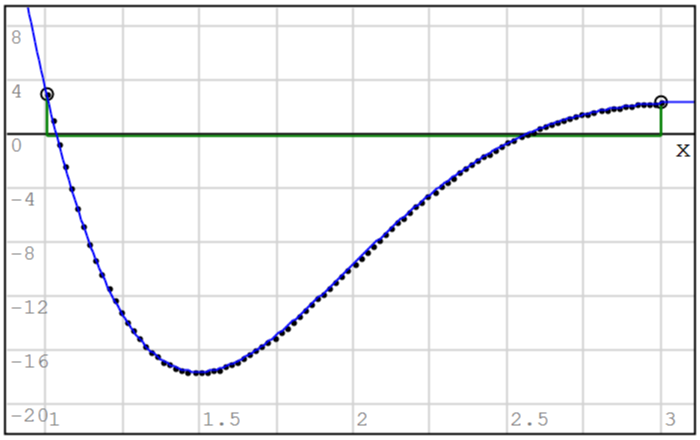
Example Solve $y'' + 3 \cdot y' + 6 = 5$ in $[a \ b] := [1 \ 3]$ subject to the boundary conditions $\begin{cases} y(1) = 3 \\ y(3) + 2 \cdot y'(3) = 5 \end{cases}$

Numeric solution $f(x) := [3 \ 6 \ 5]$ $M := \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 5 \end{bmatrix}$

Analytic solution
$$y(x) := 0.0696737 \cdot e^{-1.5 \cdot x} \cdot \left(11.9605 \cdot e^{1.5 \cdot x} + 1243.5 \cdot \sin(1.93649 \cdot x) + 2857.67 \cdot \cos(1.93649 \cdot x)\right)$$

$$y'(x) := \frac{d}{dx} y(x)$$

Results $\quad [X \;\; Y \;\; Y'] := Cols\left(lbvp_2\left(f(x), [a \;\; b], M, 100\right)\right) \quad Err = 0.21$



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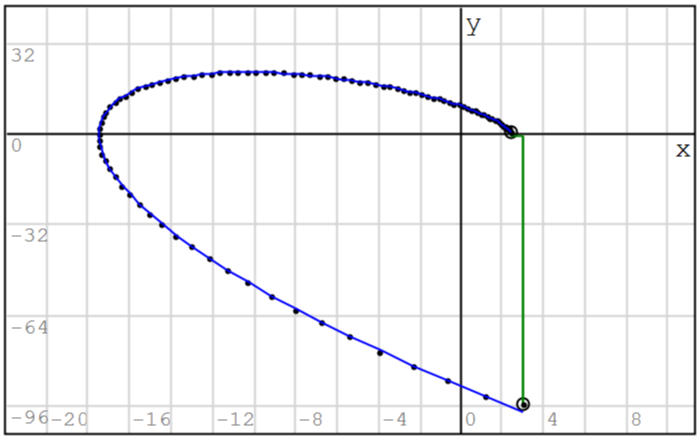


PlotYY'

$\text{Clear}\left(y(x), y'(x)\right) = 1$

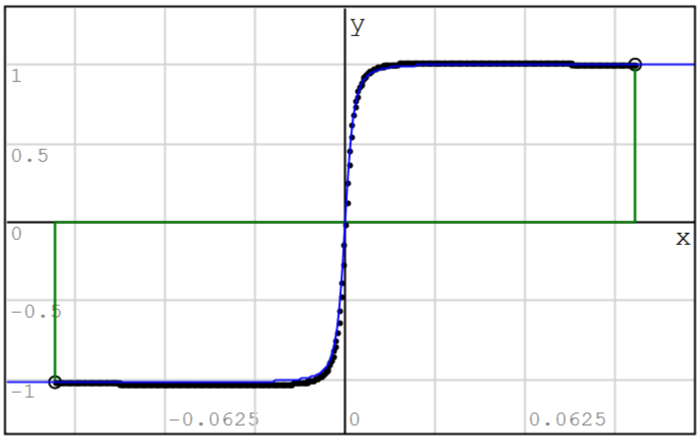
⊟—lbvp example

Example $\quad$ Solve $\quad y'' + \dfrac{3 \cdot p}{\left(p + x^2\right)^2} \cdot y = 0 \quad$ in $\quad [a \;\; b] := [-0.1 \;\; 0.1] \quad$ subject to the boundary conditions $\quad \begin{cases} y(a) = y1 \\ y(b) = y2 \end{cases} \quad$ http

Analytic solution $\quad p := 10^{-5} \qquad y(x) := \dfrac{x}{\sqrt{p + x^2}} \qquad y'(x) := \dfrac{d}{d\,x} y(x)$

Numeric solution $\quad f(x) := \left[0 \quad \dfrac{3 \cdot p}{\left(p + x^2\right)^2} \quad 0 \right] \qquad M := \begin{bmatrix} 1 & 0 & y(a) \\ 1 & 0 & y(b) \end{bmatrix}$

Results $\quad [X \;\; Y \;\; Y'] := Cols\left(lbvp_2\left(f(x), [a \;\; b], M, 500\right)\right) \quad$ **< Big N** $\quad Err = 0.29$



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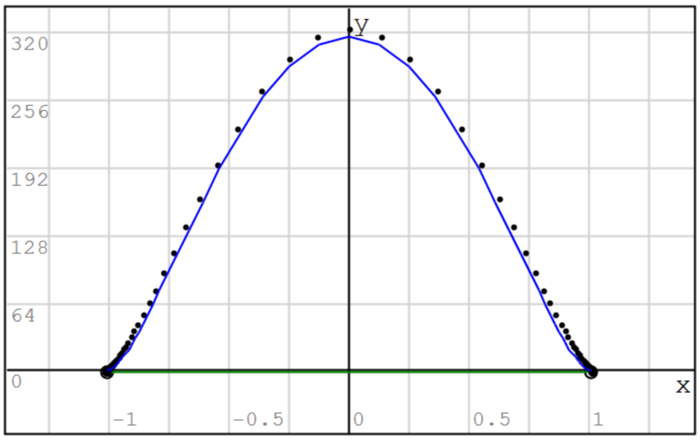


PlotYY'

$\text{Clear}\left(y(x), y'(x), p\right) = 1$

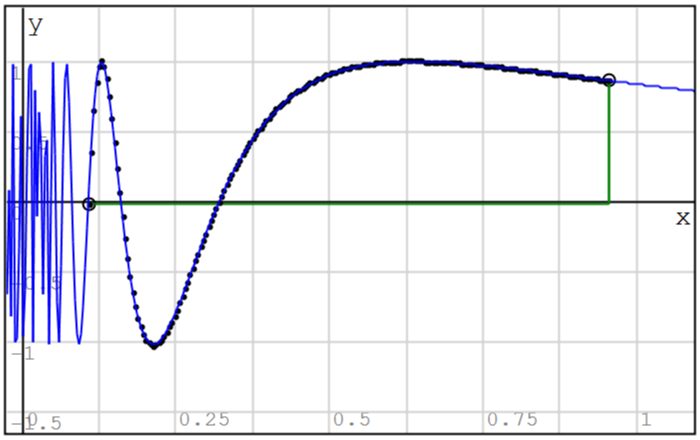
⊟—lbvp example

Example Solve $y''+\frac{2}{x}\cdot y'+\frac{y}{x^4}=0$ in $[a\ b]:=\left[\frac{1}{3\cdot\pi}\ \ \frac{3}{\pi}\right]$ subject to the boundary conditions $\begin{cases} y\left(\frac{1}{3\cdot\pi}\right)=0 \\ y\left(\frac{3}{\pi}\right)=\frac{\sqrt{3}}{2}\end{cases}$ http

Numeric solution $f(x):=\left[\frac{2}{x}\ \ \frac{1}{x^4}\ \ 0\right]$ $M:=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & \frac{\sqrt{3}}{2}\end{bmatrix}$

Analytic solution $y(x):=\sin\left(\frac{1}{x}\right)$ $y'(x):=\frac{d}{dx}y(x)$

Results $[X\ Y\ Y']:=Cols\left(lbvp_2\left(f(x),[a\ b],M,200\right)\right)$ $Err=0.11$



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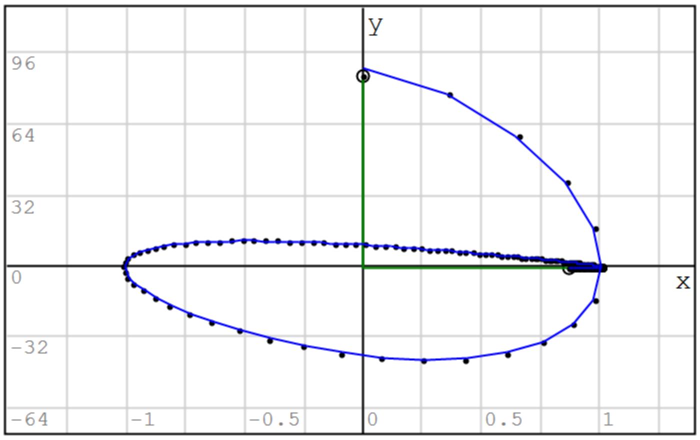


PlotYY'

$\text{Clear}(y(x),y'(x))=1$

⊟—bvp with two solutions

Example Solve $y''+|y|=0$ in $[a\ b]:=[0\ 4]$ subject to the boundary conditions $\begin{cases} y(a)=0\\ y(b)=-2\end{cases}$

This problem have two solutions. I compare the bvp solution with the shooting method, because can't found a symbolic expression.

Numeric solution $\varphi(x,y,y'):=-|y|$ $M:=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & -2\end{bmatrix}$

$[X\ Y\ Y']:=Cols\left(bvp_2\left(\varphi(x,y,y'),[a\ b],0,M,N,\varepsilon\right)\right)$

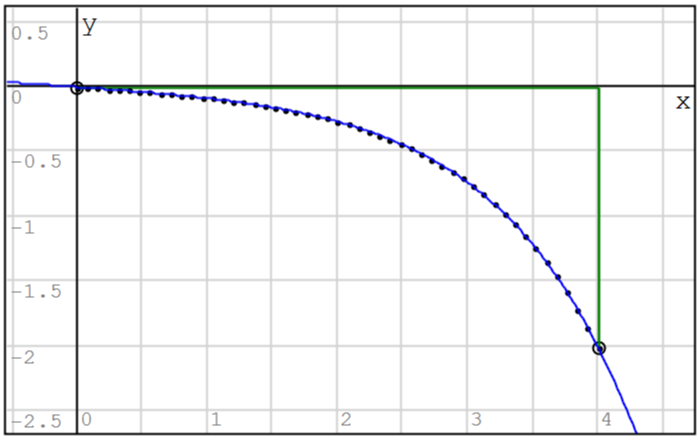
Shoot solution $[y_a\ y_b]:=\left[Y_1\ \ Y_{N+1}\right]$ Guess: $y'_a:=Y'_1$

$D(x,y):=\begin{bmatrix}y_2\\ -|y_1|\end{bmatrix}$ $sol(y'_a):=\text{al\_rkckadapt}\left(\begin{bmatrix}y_a\\ y'_a\end{bmatrix},a,b,N,D\right)$

$Eq(y'_a):=sol(y'_a)_{N+1\,2}-y_b$ $y'_a:=\text{al\_nleqsolve}\left(y'_a,Eq\right)_1=-0.0733$

$[\Xi\ \Psi\ \Psi']:=Cols\left(sol\left(y'_a\right)\right)$ $y(x):=\text{cinterp}(\Xi,\Psi,x)$ $y'(x):=\text{cinterp}(\Xi,\Psi',x)$

Results $Err = 0$



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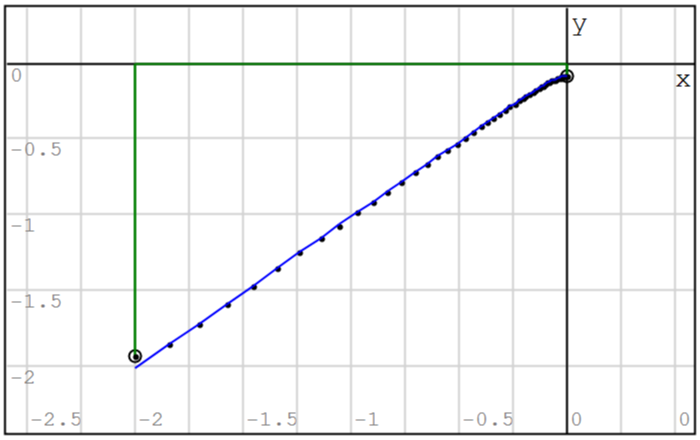


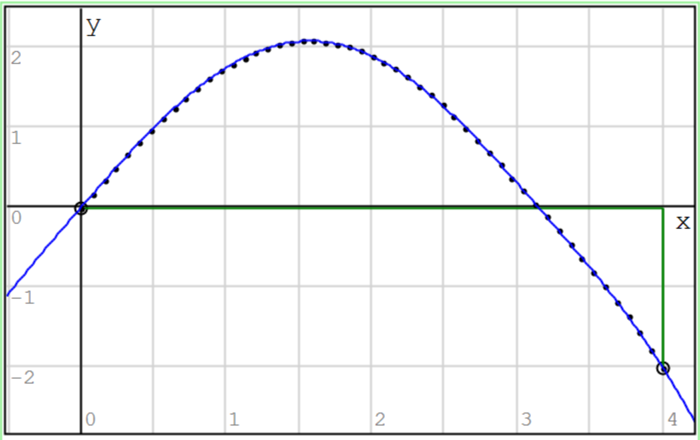
PlotYY'

For the other solution: call bvp with a new guess $\quad [X\ Y\ Y'] := Cols\left(bvp_2\left(\varphi(x, y, y'), [a\ b], Yo_{[1..(N+1)]} := 1, M, N, \varepsilon\right)\right)$

Solve the shoot with a new guess from bvp $\quad y'_a := Y'_1 \qquad y'_a := \text{al\_nleqsolve}\left(y'_a, Eq\right)_1 = 2.0666$

$[\Xi\ \Psi\ \Psi'] := Cols\left(sol\left(y'_a\right)\right) \qquad y(x) := \text{cinterp}(\Xi, \Psi, x) \qquad y'(x) := \text{cinterp}(\Xi, \Psi', x)$

Results $Err = 0.01$



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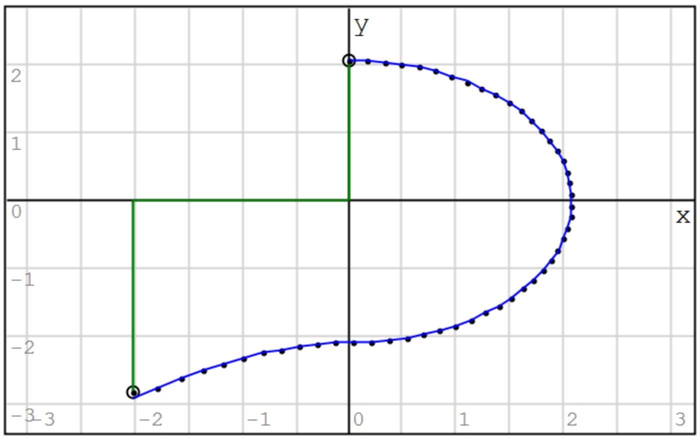


PlotYY'

$\text{Clear}(y(x), y'(x)) = 1$

Shooting method

Shooting Method

Solve this BVP using the Shooting method

$$ode := \frac{d^2}{dx^2} y(x) + \frac{1}{2} \cdot \frac{d}{dx} y(x) + y(x) - 5 \qquad [a\ b] := [-1\ 6] \qquad [y_a\ y_b] := [7\ 3]$$

Convert to a system and solve with an ode solver

$$D(x, y) := \begin{bmatrix} y_2 \\ -\frac{y_2}{2} - y_1 + 5 \end{bmatrix} \qquad N := 100 \qquad sol\left(y'_a\right) := \text{al\_rkckadapt}\left(\begin{bmatrix} y_a \\ y'_a \end{bmatrix}, a, b, N, D\right)$$

Equation to solve

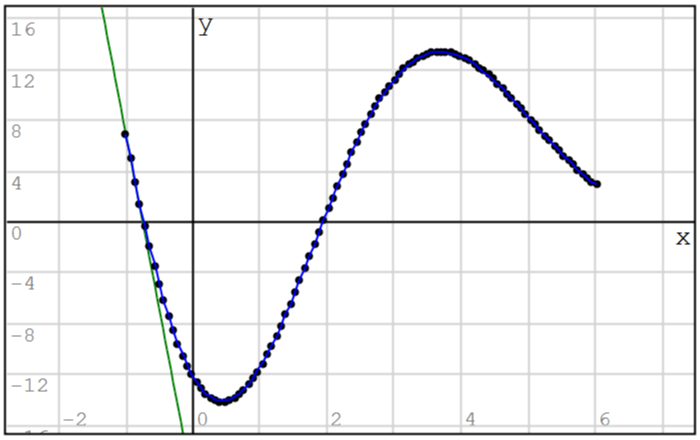
$$Eq\left(y'_a\right) := sol\left(y'_a\right)_{N+1\,2} - y_b$$

Guess for a solution $\quad y'_a := -10$

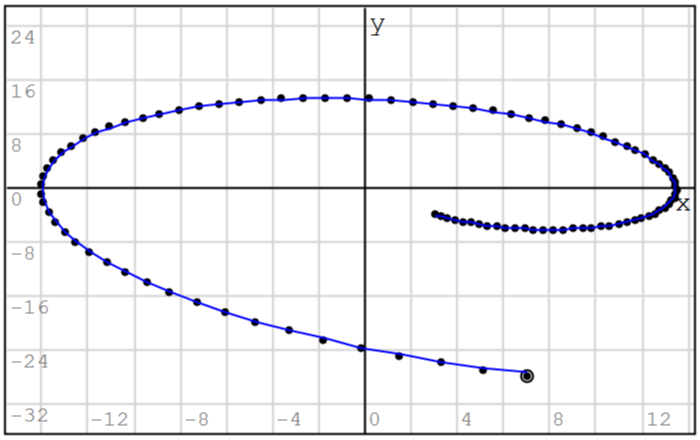
Solve $\quad y'_a := \text{al\_nleqsolve}\left(y'_a, Eq\right)_1 = -27.5704 \qquad [X\ Y\ Y'] := Cols\left(sol\left(y'_a\right)\right)$

lbvp solution $\quad f(x) := \begin{bmatrix} \frac{1}{2} & 1 & 5 \end{bmatrix} \qquad M := \begin{bmatrix} 1 & 0 & 7 \\ 1 & 0 & 3 \end{bmatrix} \qquad [\Xi\ \Psi\ \Psi'] := Cols\left(lbvp_2\left(f(x), [a\ b], M, N\right)\right)$

Results $\qquad \text{norme}(Y - \Psi) = 0.34$



$\begin{cases} \text{augment}(\Xi, \Psi) \\ \text{augment}(X, Y, ".") \\ y'_a \cdot (x - a) + y_a \end{cases}$



$\begin{cases} \text{augment}(\Psi, \Psi') \\ \text{augment}(Y, Y', ".") \\ \text{augment}(y_a, y'_a, "o") \end{cases}$

Alvaro