# Response Analysis of Ladder Beams Under Inertial Moving Load

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# Response Analysis of Ladder Beams Under Inertial Moving Load

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With the continuous development of industry, variable-section beams and high speed moving loads with large mass are widely used. Thus, it is of great significance to study the vibration response of variable-section beam with the consideration of inertia effect. Most past research focuses on the vibration response on uniform beams considering inertial effects, but there is little research on the vibration response of moving loads on variable section beam considering the inertia effect. In this paper, a variable section beam is simplified as a multi-stage ladder beam. Using the Euler-Bernoulli beam model, free-vibration characteristics and forced vibration characteristics of cantilever ladder beam are analysed. Following this step the vibration response considering the influence of the inertia effect is studied and compared with the situation that does not consider the influence of inertia effect. The results show that the mass, velocity, and acceleration of moving loads influence the effect of inertia on the response. Mass is the main factor affecting the results. The inertia effect caused by the acceleration and velocity can be ignored when the mass of moving load is small. The results have good engineering applicability.

## NOMENCLATURE

| | |
|---|---|
| $a$ | Lateral acceleration due to inertial effects. |
| $acc$ | Lateral acceleration of beam. |
| $A$ | Cross-sectional area of beam. |
| $[C]$ | Generalized damping matrix. |
| $d$ | Inner diameter. |
| $D$ | Outer diameter. |
| $dis$ | Lateral displacement of beam (In numerical calculation). |
| $E$ | Elastic modulus. |
| $[F]$ | Generalized matrix of forces. |
| $g$ | Acceleration of gravity. |
| $[H]$ | Transfer matrix. |
| $H_m$ | Dynamic response considering inertial effect. |
| $H_n$ | Dynamic response without considering inertial effect. |
| $i$ | The $i$-th cross section of beam. |
| $[I]$ | Unit matrix. |
| $j$ | The $j$-th natural frequency. |
| $k$ | Intermediate variable. |
| $[K]$ | Generalized stiffness matrix. |
| $L$ | Length of beam. |
| $Lap$ | Laplace transform. |
| $M$ | Mass of moving load. |
| $[M]$ | Generalized mass matrix. |
| $\rho$ | Density of beam. |
| $q$ | Generalized coordinate. |
| $[Q]$ | Generalized displacement matrix of beam. |
| $R$ | Mode ratio. |
| $s$ | Variable in Laplace transform. |
| $t$ | Time variable. |
| $\mu$ | Displacement of moving load. |
| $V$ | Velocity of moving load. |
| $vel$ | Lateral velocity of beam. |
| $w$ | Circular frequency. |
| $y$ | Lateral displacement of beam (In vibration formula). |
| $Y$ | Laplace transform function. |
| $\lambda$ | Inertia influence coefficient. |
| $\overline{\phi}$ | Regular modal function. |
| $\phi$ | Modal function. |
| $\theta$ | Rotation of the beam. |
| $[\eta]$ | Connection matrix. |

## 1. INTRODUCTION

The dynamic problems of variable cross-section beams under moving loads are very common in engineering practice, such as vehicle-bridge coupling vibration, track vibration, projectile barrel coupling vibration, and fluid-solid coupling vibration. With the development of modern industry, the application of high-speed, large-acceleration, and large-scale moving components and complex structures in engineering are increasing. It is of great significance to analyse the vibration response of variable cross-section beams and to consider the influence of inertial effects of moving loads on vibration response.

Dynamic analysis of beams under moving loads has always been an important issue in structural engineering. A lot of work on the vibration of beams under moving loads has been reported.[1,2] Museros and Moliner[3] studied the vibration of simply supported beams under a constant moving load and proposed a new approximation method to estimate the maximum acceleration. Using modal superposition, Sudheesh Kumar[4] proposed a simple and compact formula to determine the free-vibration response of a uniform beam under a single moving load. Using Laplace transform, Johansson[5] obtained a closed solution to the vibration response of the Euler-Bernoulli beam under constant moving loads. X Wang[6] analysed the dynamic behaviour of functionally graded material (FGM) beams under a moving point load. Using the Euler-Bernoulli beam hypothesis, Dimitrovoá[7] obtained a new formula for the critical velocity of a uniformly moving load. It is assumed that the load is traversing a beam supported by a foundation of a finite depth. Simplified plane models of the foundation are presented for the finite and infinite beams. Based on the Euler-Bernoulli beam hypothesis and von Kármán geometric nonlinear theory, Tao[8] studied the nonlinear dynamic behaviours of fibre metal laminated beams under moving loads in thermal environments. Based on the Biot consolidation theory and the Timoshenko beam model, Keivan[9] analysed the dynamic response of porous elastic beams under and with consideration to the moving point of shear deformation. In consideration of Poisson's effect, shear deformation, and rotary inertia, Mohammad H. Kargamovin[10] analysed the dynamic response of a delaminated composite beam under the action of moving oscillatory mass. HB Khaniki[11] analysed the dynamic behaviour of multi-layered viscoelastic nanobeams resting on a viscoelastic medium with a moving nanoparticle. The influences of the nonlocal parameter, stiffness and damping parameter of medium, internal damping parameter, and number of layers are studied while the nanoparticle passes through. Based on the modified couple stress theory, the dynamic behaviour of multilayered microbeam systems with respect to a moving load is analysed.[12]

In research on non-uniform beams, HB Khaniki analysed the mechanical behaviour of non-uniform small scale beams in the framework of nonlocal strain gradient theory,[13] and concluded that having a non-uniform cross section in nonlocal strain gradient beams could lead to significant changes in mechanical behaviour of such structures. Khaniki[14] analysed non-uniformity effects on free vibration analysis of functionally graded beams. Jiang and Bert[15] studied the vibration behaviour of step beams under various boundary conditions. In this case, the frequency can be derived from a fourth-order determinant equal to zero, from which the free-vibration mode can be solved. Naguleswaran[16,17] successfully derived the more complex geometry of the beam vibration modes. Dong[18] analysed the vibration characteristics of the stepped beam using the Timoshenko beam model, including the effects of shear deformation and rotational inertia. Wu and Hsu[19] developed a numerical procedure that can effectively be used to calculate the natural frequency and mode shapes when considering the mass-concentration beam vibrations. Lu[20] developed a composite element method that combines the finite element method with the classical beam theory and proved to be correct, introducing the free and forced vibrations of beams with either single or multiple-step changes using the composite element method (CEM). Using a numerical program proposed by Xu,[21] some existing numerical problems that arise in the calculation of the high-order mode of the ladder beam are avoided. The lowest three natural frequencies of a multistep up and down cantilever beam using a global Rayleigh-Ritz formulation, component modal analyses (CMA), ANSYS, and experimental are evaluated.[22] Adomian decomposition method (ADM) was used to obtain the effect of step ratio and step location on a beam's natural frequencies.[23,24] The free and forced vibrations of beams with either single or multiple-step changes using the composite element method (CEM) were introduced by Lu.[25]

It can be seen that although researchers have deeply studied the dynamic response of moving loads on uniform beams and the free-vibration characteristics of stepped beams, few studies have investigated the dynamic response of moving load on stepped beams considering the inertial effects. In this paper, the response of moving load a on ladder beam considering the inertial effect is analysed and compared with the vibration response of moving load without considering the inertial effect. The factors that affect the inertial effect are also discussed.

## 2. THE NATURAL FREQUENCY AND MODE SHAPE FUNCTION OF LADDER CANTILEVER BEAM

When analysing the vibration characteristics of variable cross-section beams, the analysis may be complicated due to the irregularity of the cross-section structure making it very difficult to obtain accurate analytical solutions. The traditional method of simplifying the variable cross-section beam into an equivalent cross-section beam may produce large errors. However, if the variable section beam is simplified as a multi-step ladder beam, the result will be calculated with great precision and the simplification error will be reduced. Simplified form is shown in the figure below.

Therefore, it is of great significance to analyse the vibration characteristics of a stepped beam. This paper takes the three-step cantilever beam as an example to analyse the free vibration characteristics and forced vibration characteristics.

The model analysed in this paper is as follows:

From the theory of vibration mechanics, the lateral free vibration equation of each section is:

$$\frac{\partial^2\left(EI_i\frac{\partial^2 y(x,t)}{\partial x^2}\right)}{\partial x^2}+\rho A_i\frac{\partial^2 y(x,t)}{\partial t^2}=0,\quad i=1,2,3. \tag{1}$$

Assuming the whole system performs harmonic vibration at the same frequency, separation of variables are used to separate

$$y(x,t)=\phi_i(x)\sin(wt). \tag{2}$$

Substituting Eq. (2) into Eq. (1) gives

$$\phi_i^4(x)-k_i^4\phi_i(x)=0; \tag{3}$$

$$k_i^4=\frac{\rho A_i}{EI_i}\omega^2. \tag{4}$$

Laplace transform the above formula to get:

$$Lap\left[\phi_i(x)\right]=\frac{s^3Y(0)+s^2Y'(0)+sY''(0)+Y'''(0)}{s^4-k_i^4}. \tag{5}$$

Inverse Laplace transform both sides of Eq. (5) and combine like terms to get:

$$\phi_i(x)=\phi_i(0)S(k_ix)+k_i^{-1}\phi_i'(0)T(k_ix)+ \\ k_i^{-2}\phi_i''(0)U(k_ix)+k_i^{-3}\phi_i'''(0)V(k_ix). \tag{6}$$

The $S$, $T$, $U$, $V$ are on behalf of the Krylov Function. The expressions are as follows:

$$S(k_ix)=\frac{1}{2}\left(\cos h(k_ix)+\cos(k_ix)\right); \tag{7a}$$

$$T(k_ix)=\frac{1}{2}\left(\sin h(k_ix)+\sin(k_ix)\right); \tag{7b}$$

$$U(k_ix)=\frac{1}{2}\left(\cos h(k_ix)-\cos(k_ix)\right); \tag{7c}$$

$$V(k_ix) = \frac{1}{2}\left(\sin h(k_ix) - \sin(k_ix)\right). \quad (7d)$$

The relationship between the left and right ends of i-th section of the ladder beam can be expressed by Eqs. (6) and (7).The derivation of Eq. (6) are as follows:

$$\phi_i'(x) = k_i\phi_i(0)V(k_ix_i) + \phi_i'(0)S(k_ix_i) + k_i^{-1}\phi_i''(0)T(k_ix_i) + k_i^{-2}\phi_i'''(0)U(k_ix_i); \quad (8a)$$

$$\phi_i''(x) = k_i^2\phi_i(0)U(k_ix_i) + k_i\phi_i(0)V(k_ix_i) + \phi_i''(0)S(k_ix_i) + k_i^{-1}\phi_i'''(0)T(k_ix_i); \quad (8b)$$

$$\phi_i'''(x) = k_i^3\phi_i(0)T(k_ix_i) + k_i^2\phi_i'(0)U(k_ix_i) + k_i\phi_i''(0)V(k_ix_i) + \phi_i'''(0)S(k_ix_i). \quad (8c)$$

The boundary conditions of the left and right ends of the i-th section can be obtained by combining Eqs. (6), (7), and (8), which can be expressed in matrix form:

$$\begin{bmatrix}\phi_i(L_i)\\ \phi_i'(L_i)\\ \phi_i''(L_i)\\ \phi_i'''(L_i)\end{bmatrix} = H\left[k_iL_i\right]\begin{bmatrix}\phi_i(0)\\ \phi_i'(0)\\ \phi_i''(0)\\ \phi_i'''(0)\end{bmatrix}. \quad (9)$$

The matrix form of $H[k_iL_i]$ is as follows:

$$H[k_iL_i] = \begin{bmatrix} S(k_iL_i) & \frac{T(k_iL_i)}{k_i} & \frac{U(k_iL_i)}{k_i^2} & \frac{V(k_iL_i)}{k_i^3} \\ V(k_iL_i)k_i & S(k_iL_i) & \frac{T(k_iL_i)}{k_i} & \frac{U(k_iL_i)}{k_i^2} \\ U(k_iL_i)k_i^2 & V(k_iL_i)k_i & S(k_iL_i) & \frac{T(k_iL_i)}{k_i} \\ T(k_iL_i)k_i^3 & U(k_iL_i)k_i^2 & V(k_iL_i)k_i & S(k_iL_i)\end{bmatrix}. \quad (10)$$

At the connection part between each beam, displacement, rotation angle, moment, and shear satisfy the condition of displacement continuity and compatibility of stress as shown by:

$$\phi_{i-1}(L_{i-1}) = \phi_i(0); \quad (11a)$$

$$\phi_{i-1}'(L_{i-1}) = \phi_i'(0); \quad (11b)$$

$$EI_{i-1}\phi_{i-1}''(L_{i-1}) = EI_i\phi_i''(0); \quad (11c)$$

$$EI_{i-1}\phi_{i-1}'''(L_{i-1}) = EI_i\phi_i'''(0). \quad (11d)$$

The connection matrix of each segment of the ladder beam is:

$$[\eta_i] = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{I_{i-1}}{I_i} & 0\\ 0 & 0 & 0 & \frac{I_{i-1}}{I_i}\end{bmatrix}. \quad (12)$$

Through the above analysis and derivation, the whole transfer matrix of the ladder beam can be obtained:

$$\begin{bmatrix}\phi_i(L_i)\\ \phi_i'(L_i)\\ \phi_i''(L_i)\\ \phi_i'''(L_i)\end{bmatrix} = H[k_iL_i][\eta_i]H[k_{i-1}L_{i-1}][\eta_{i-1}]\cdots \cdots H[k_1L_1]\begin{bmatrix}\phi_1(0)\\ \phi_1'(0)\\ \phi_1''(0)\\ \phi_1'''(0)\end{bmatrix}. \quad (13)$$

In this paper, taking the three-step cantilever beam as an example, $i = 3$ in the above formula. The boundary condition of the cantilever beam is that the left end which is the fixed end and the right end which is the free end. The displacement and the rotation angle of the fixed end are 0, the moment and the shear of the free end are 0:

$$\phi_3''(L_3) = 0; \quad (14a)$$

$$\phi_3'''(L_3) = 0; \quad (14b)$$

$$\phi_1(0) = 0; \quad (14c)$$

$$\phi_1'(0) = 0. \quad (14d)$$

The matrix $[Q]$ is defined as:

$$[Q] = H[k_iL_i][\eta_i]H[k_{i-1}L_{i-1}][\eta_{i-1}]\cdots H[k_1L_1]. \quad (15)$$

From which one can derive:

$$\begin{bmatrix}\phi_3(L_3)\\ \phi_3'(L_3)\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & Q_{13} & Q_{14}\\ Q_{21} & Q_{22} & Q_{23} & Q_{24}\\ Q_{31} & Q_{32} & Q_{33} & Q_{34}\\ Q_{41} & Q_{42} & Q_{43} & Q_{44}\end{bmatrix}\begin{bmatrix}0\\ 0\\ \phi_1''(0)\\ \phi_1'''(0)\end{bmatrix}. \quad (16)$$

It can be derived from the theory of linear algebra:

$$\begin{bmatrix}Q_{33} & Q_{34}\\ Q_{43} & Q_{44}\end{bmatrix}\begin{bmatrix}\phi_1''(0)\\ \phi_1'''(0)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}. \quad (17)$$

Since the $\phi_1''(0)$ and $\phi_1'''(0)$ could be any value, the determinant of matrix $[Q]$ in Eq. (17) is 0:

$$\begin{vmatrix}Q_{33} & Q_{34}\\ Q_{43} & Q_{44}\end{vmatrix} = 0. \quad (18)$$

The circular frequency $w_j$ is the only variable in $[Q]$. Calculating the above determinant, the circular frequency $w_j$ can be obtained. Substituting $w_j$ into Eq. (15) to get the corresponding $[Q]$ Substituting $[Q]$ into Eq. (18), the ratio of $\phi_{1j}''(0)$ and $\phi_{1j}'''(0)$ is obtained as:

$$R_j = \frac{\phi_{1j}''(0)}{\phi_{1j}'''(0)}. \quad (19)$$

Substituting the $R_j$ into Eq. (16), the shape function of the third section beam can be obtained as:

$$\phi_{3j} = Q_{13}R_j + Q_{14}. \quad (20)$$

The mode shape functions of different sections of beams are different. By this method, the mode shape functions of each beam can be obtained. Because of space limitations, not all segments are illustrated. Using mode shape function to solve the generalized mass and the generalized stiffness, from the orthogonality of mode shape function:

$$M_j = \sum_{i=1}^{3}\int_{l_{i-1}}^{l_i}\rho A_i\phi_{ij}^2(x)dx = \int_0^{l_1}\rho A_1\phi_{1j}^2(x)dx + \int_{l_1}^{l_2}\rho A_2\phi_{2j}^2(x)dx + \int_{l_2}^{l_3}\rho A_3\phi_{3j}^2(x)dx; \quad (21a)$$

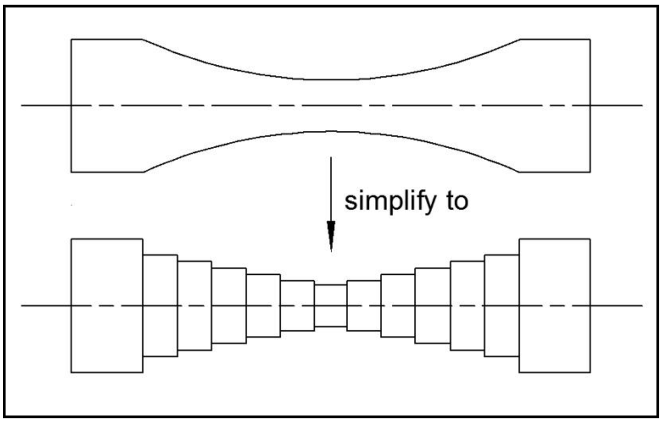


**Figure 1.** Variable section beam simplifies to multiple-step beam.

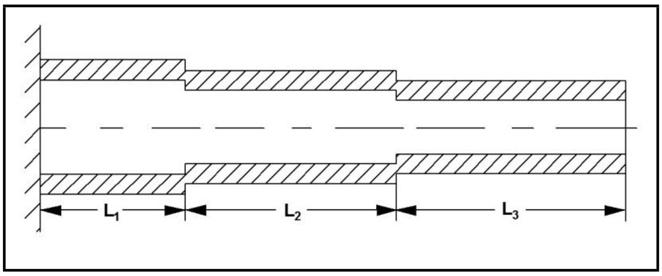


**Figure 2.** Three-step cantilever beam.

$$K_j = \sum_{i=1}^{3} \int_{l_{i-1}}^{l_i} EI_i {\phi''_{ij}}^2(x)dx = \int_{0}^{l_1} EI_1 {\phi''_{1j}}^2(x)dx + \int_{l_1}^{l_2} EI_2 {\phi''_{2j}}^2(x)dx + \int_{l_2}^{l_3} EI_3 {\phi''_{3j}}^2(x)dx. \quad (21b)$$

The generalized mass and generalized stiffness are expressed in matrix form:

$$[M] = \begin{bmatrix} M_1 & 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 & 0 \\ 0 & 0 & M_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & M_j \end{bmatrix}; \quad (22)$$

$$[K] = \begin{bmatrix} K_1 & 0 & 0 & 0 & 0 \\ 0 & K_2 & 0 & 0 & 0 \\ 0 & 0 & K_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & K_j \end{bmatrix}; \quad (23)$$

and the regular modal function is obtained:

$$\overline{\phi_{ij}} = \frac{\phi_{ij}}{\sqrt{M_j}}. \quad (24)$$

This method can solve the natural frequency and mode shape function of the stepped beam with arbitrary boundary conditions.

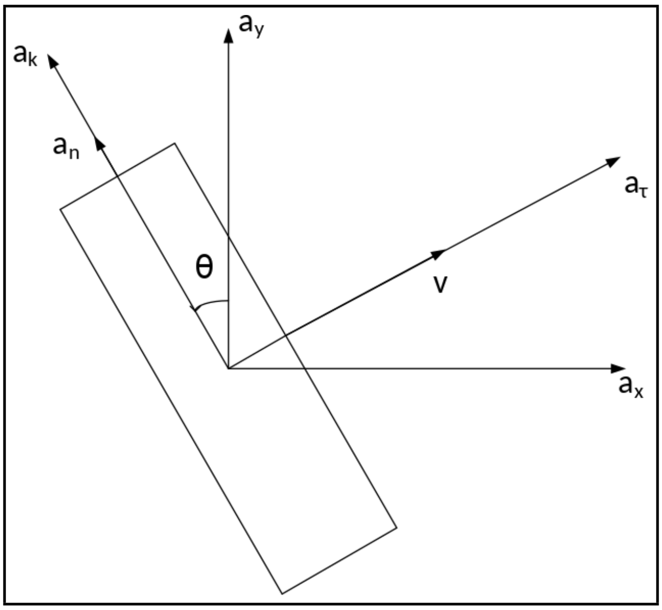


**Figure 3.** Axial movement of beam elements.

## 3. FORCED VIBRATION REPONSE CALCULATION OF STEPPED BEAM UNDER MOVING LOAD WITH INERTIAL EFFECT

When considering the inertia effect of moving load acting on the beam, select a micro-segment of the bending beam, the movement state is as follows:

It can be seen from Fig. 3, $\theta$ is the angle of the cross section; $v$ is the velocity of the beam moving in the axial direction; $a_y$ is the lateral acceleration of the cross section due to the bending deformation of the beam; $a_x$ is the lateral acceleration of the cross section due to the bending deformation of the beam; $a_\tau$ is the tangential acceleration of the cross section moving in the axial direction; $a_n$ is the normal acceleration of the cross section moving in the axial direction; and $a_k$ is the Coriolis acceleration caused by the interaction between rotational angular velocity and axial motion velocity.

According to the knowledge of material mechanics and theoretical mechanics:

$$\theta = \frac{\partial y}{\partial x}; \quad (25a)$$

$$a_y = \frac{\partial^2 y}{\partial t^2}; \quad (25b)$$

$$a_\tau = \frac{dv}{dt}; \quad (25c)$$

$$a_n = v^2 \frac{\partial^2 y}{\partial x^2}; \quad (25d)$$

$$a_k = 2v\frac{\partial \theta}{\partial t} = 2v\frac{\partial^2 y}{\partial x \partial y}. \quad (25e)$$

The actual lateral acceleration of the cross section is as follows:

$$\begin{aligned} a &= a_y + a_\tau \sin\theta + a_n \cos\theta + a_k \cos\theta \\ &= a_y + a_\tau \theta + a_n + a_k \\ &= \frac{\partial^2 y}{\partial t^2} + \frac{dv}{dt}\frac{\partial y}{\partial x} + v^2\frac{\partial^2 y}{\partial x^2} + 2v\frac{\partial^2 y}{\partial x \partial t}. \end{aligned} \quad (26)$$

Considering the influence of gravity acceleration, the actual force of the moving load is:

$$P(x,t) = m \cdot \left\{ g - \left[ \frac{\partial^2 y}{\partial t^2} + 2\frac{\partial^2 y}{\partial x \partial t}\dot{\mu}(t) + \frac{\partial^2 y}{\partial x^2}\dot{\mu}^2(t) + \frac{\partial y}{\partial x}\ddot{\mu}(t) \right] \right\}. \quad (27)$$

Substituting Eq. (27) into the lateral vibration equation of the ladder beam:

$$\frac{\partial^2 \left( EI_i \frac{\partial^2 y(x,t)}{\partial x^2} \right)}{\partial x^2} + \rho A_i \frac{\partial^2 y(x,t)}{\partial t^2} = P(x,t)\delta(x-\mu(t)), \quad i = 1,2,3; \quad (28)$$

where $\delta(x-\mu(t))$ is the Dirichlet function, it is known that:

$$\begin{cases} x = \mu(t), \delta(x-\mu(t)) = 1 \\ x \neq \mu(t), \delta(x-\mu(t)) = 0 \end{cases}. \quad (29)$$

Using the method of separation of variables and the orthogonality of modal functions, substitute Eqs. (2), (21), and (24) for:

$$\ddot{q}_{ij}(t) + w_j^2 q_{ij}(t) = m \left\{ g - \sum_{n=1}^{\infty} [2\dot{\mu}(t)\phi'_{in}(\xi)\dot{q}_{in}(t) + \dot{\mu}^2(t)\phi''_{in}(\xi)q_{in}(t) + \ddot{\mu}(t)\phi'_{in}(\xi)q_{in}(t) + \phi_{in}(\xi)\ddot{q}_{in}(t)] \right\} \overline{\phi}_{ij}(\xi). \quad (30)$$

In Eq. (30), $i$ represents the *i-th* section and $j$ represents the *j-th* natural frequency. Substituting Eq. (8), the above formula is expressed in matrix form:

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = [F]. \quad (31)$$

among them being:

$$[I] = diag(1, 1, \ldots)_j; \quad (32a)$$

$$[\phi_{ij}] = [\phi_{j1}(\xi), \phi_{j2}(\xi), \phi_{j3}(\xi), \phi_{j4}(\xi), \ldots]^T; \quad (32b)$$

$$[M] = [I] + m \cdot [\phi_{ij}(\xi)] \cdot [\phi_{ij}(\xi)]^T; \quad (32c)$$

$$[C] = 2m \cdot \dot{\mu}(t)[\phi_{ij}(\xi)][\phi'_{ij}(\xi)]^T; \quad (32d)$$

$$[K] = diag[w_j^2] + m \cdot \ddot{\mu}(t) \cdot [\phi_{ij}(\xi)][\phi'_{ij}(\xi)]^T + m \cdot \dot{\mu}^2(t) \cdot [\phi_{ij}(\xi)][\phi''_{ij}(\xi)]^T; \quad (32e)$$

$$[F] = m \cdot g \cdot [\phi_{ij}(\xi)]. \quad (32f)$$

Loads with inertia effects form an additional generalized mass matrix, generalized stiffness matrix, and generalized damping matrix. As the force is a time-varying force, the additional matrix is time-varying, so it is difficult to get analytic solution. $\mu(t)$ is a function of $t$ and $\phi_{ij}(\xi)$ is a function of $\xi$ that can be transformed into a function of $t$. In this paper, the Newmark algorithm is used to calculate the response. In order to simplify the calculation process, on the premise of the accuracy of calculations, use the modal truncation method to intercept the first four natural frequencies for analysis. Using the continuity of displacement, velocity, and acceleration, the forced response is obtained as:

$$dis_{i-1}(L_{i-1}) = dis_i(0); \quad (33a)$$

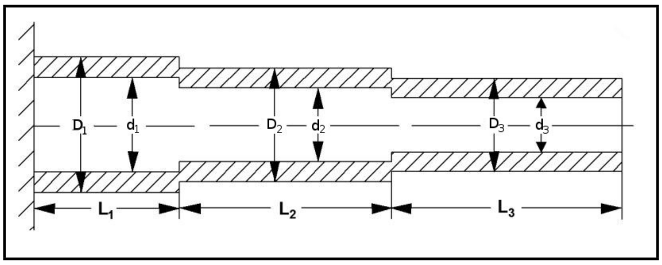


**Figure 4.** The model to be analysed.

$$vel_{i-1}(L_{i-1}) = vel_i(0); \quad (33b)$$

$$acc_{i-1}(L_{i-1}) = acc_i(0). \quad (33c)$$

where $dis_{i-1}(L_{i-1})$ stands for the displacement of the right end of the $(i-1)$-*th* section, and $dis_i(0)$ stands for the displacement of the left end of the *i-th* section. Velocity and acceleration are also used in similar expressions.

Define the inertia influence coefficient using:

$$\lambda = \left| \frac{H_m - H_n}{H_n} \right| \times 100\%; \quad (34)$$

where $\lambda$ is inertia influence coefficient, $H_m$ is dynamic response considering inertial effect, and $H_n$ is dynamic response without considering inertial effect.

The effect of inertia effect on vibration response can be analysed by judging the numerical magnitude of $\lambda$.

## 4. EXAMPLE ANALYSIS AND DISCUSSION

### 4.1. Calculation of The Natural Frequencies and Modes Of The Ladder Cantilever Beam

Substituting specific values, the elastic modulus $E = 2.1 * 10^{11}$ Pa, density $\rho = 7800$ kg/m$^3$. The model for be analysis is shown in Fig. 4.

After changing the inner diameter, outer diameter, and length of the stepped beam, calculating the natural frequencies, and comparing with the results of ANSYS, the overall results are shown in Table 1.

The results calculated using the transfer matrix method are similar to those calculated by the finite element method. The relative error of the first natural frequency is smaller than 0.3%, the biggest error is less than 5 %. The results show the accuracy of the calculation using the transfer matrix method.

ANSYS uses the finite-element method to calculate the natural frequency of stepped beam. While the stepped beam is considered as a continuum in theoretical calculation, due to different calculation methods, the results have deviation. The accuracy of ANSYS meshing may also affect the calculation results.

Substituting $D1 = 0.4$ m, $d1 = 0.16$ m, $D2 = 0.2$ m, $d2 = 0.1$ m, $D3 = 0.14$ m and $d3 = 0.04$ m, the shape function is as shown in Fig. 5.

The vibration mode function image of the stepped cantilever beam is very similarto the vibration mode function image of constant section beam. When the mode shape function is close to the free end, the "steepness" of the mode shape's function increases. This is because the section modulus of the second beam and the third beam are relatively small in this example.

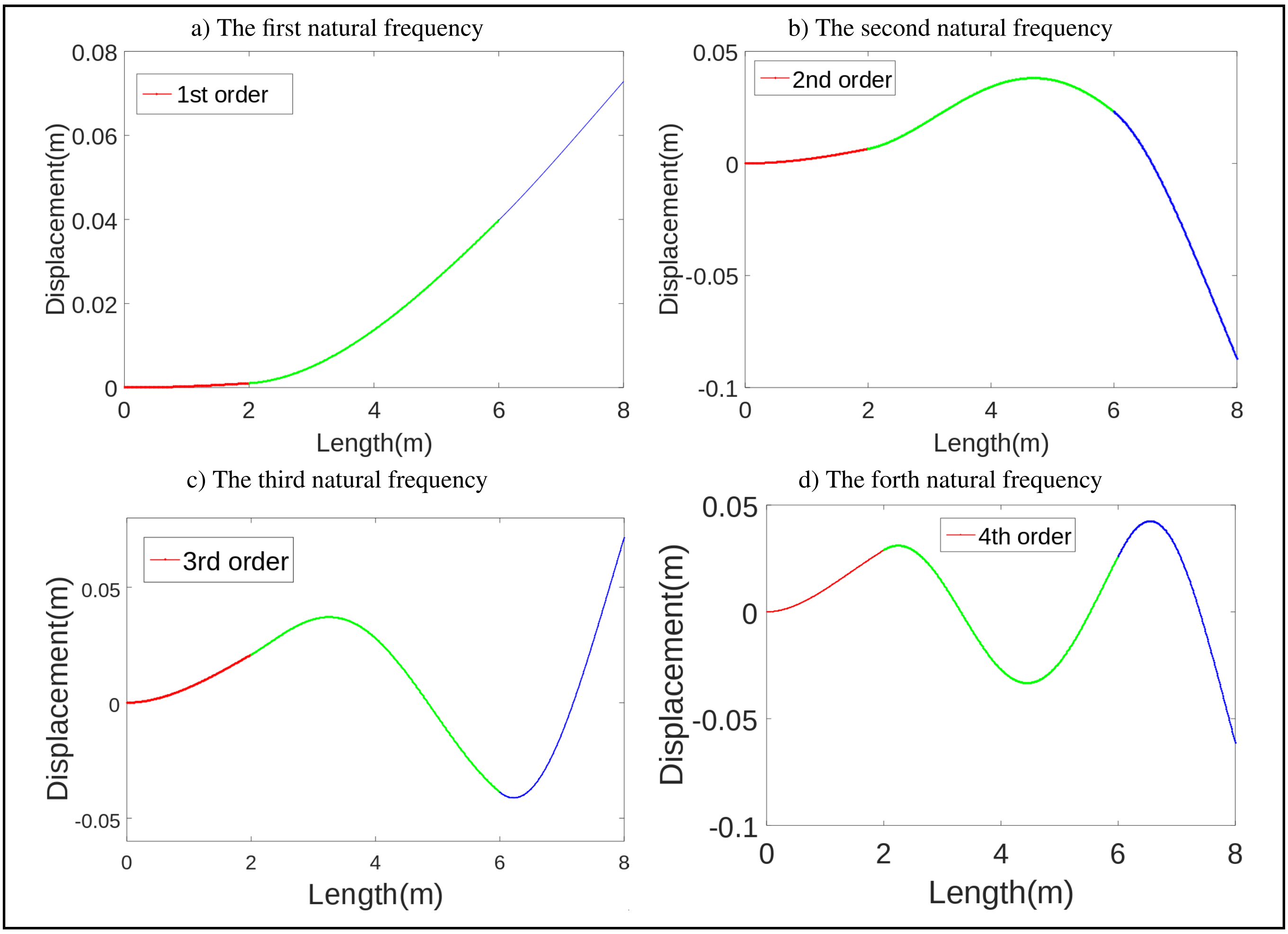


**Figure 5.** The first four mode shapes of the stepped cantilever beam.

**Table 1.** Natural frequency of cantilevered ladder beam.

| Length and diameter (m) | The result of MATLAB (Hz) | The result of ANSYS (Hz) | Relative error (%) |
|---|---|---|---|
| D1=0.40; d1=0.16; D2=0.20; d2=0.10; D3=0.14; d3=0.04; L1=2.00; L2=4.00; L3=2.00. | 5.12 | 5.11 | 0.20 |
| | 23.88 | 27.73 | 0.63 |
| | 51.87 | 51.14 | 1.43 |
| | 85.89 | 83.93 | 2.34 |
| D1=0.45; d1=0.18; D2=0.25; d2=0.12; D3=0.20; d3=0.06; L1=2.00; L2=4.00; L3=2.00. | 5.68 | 5.67 | 0.23 |
| | 28.78 | 28.49 | 1.00 |
| | 63.63 | 62.35 | 2.05 |
| | 108.32 | 104.95 | 3.21 |
| D1=0.50; d1=0.20; D2=0.30; d2=0.14; D3=0.24; d3=0.08; L1=2.00; L2=4.00; L3=2.00. | 6.67 | 6.65 | 0.29 |
| | 33.04 | 32.60 | 1.34 |
| | 73.16 | 71.29 | 2.63 |
| | 128.80 | 123.51 | 4.28 |
| D1=0.40; d1=0.16; D2=0.20; d2=0.10; D3=0.14; d3=0.04; L1=2.00; L2=3.00; L3=2.00. | 7.31 | 7.29 | 0.23 |
| | 30.59 | 30.35 | 0.80 |
| | 63.43 | 62.24 | 1.91 |
| | 111.30 | 108.32 | 2.75 |
| | 10.95 | 10.92 | 0.29 |
| | 40.35 | 39.91 | 1.11 |
| | 79.20 | 77.34 | 2.40 |
| | 156.72 | 151.64 | 3.35 |

## 4.2. The Influence of Inertia on Vibration Response

Newark method is used to calculate the response. In the calculation $\alpha = 0.8$, $\beta = 0.8$, the gravitational acceleration $g = 10$, change the mass, velocity, and acceleration of the moving load to analyse the change of inertial influence coefficient.

### 4.2.1. The Impact of the Mass and Velocity of Moving Load on Response Results

Assume that the moving load performs a uniform motion on the beam. Change the mass and velocity of the moving load. When the moving load moves to the free end, record the value of the lateral displacement of the free end at this moment. The results are shown in Table 2.

**Table 2.** Comparison of response results considering different masses and velocities of moving load.

| $L_1$= 2 m, $L_2$= 4 m, $L_3$= 2 m, $D_1$= 0.4 m, $d_1$= 0.16 m, $D_2$= 0.2 m, $d_2$= 0.1 m, $D_3$= 0.14 m, $d_3$= 0.04 m | | | | |
|---|---|---|---|---|
| $V$ (m/s) | $M$ (kg) | Displacement/ mm: Only considering gravity | Displacement/ mm: Considering inertia effect | Inertial influence coefficient $\lambda$ |
| 10 | 5 | -0.2790 | -0.2789 | 0.05% |
| | 20 | -1.1162 | -1.1138 | 0.21% |
| | 50 | -2.7904 | -2.7730 | 0.62% |
| | 100 | -5.5808 | -5.4973 | 1.50% |
| 20 | 5 | -0.2677 | -0.2669 | 0.31% |
| | 20 | -1.0708 | -1.0581 | 1.19% |
| | 50 | -2.6770 | -2.6036 | 2.74% |
| | 100 | -5.3540 | -5.0918 | 4.90% |
| 30 | 5 | -0.2889 | -0.2876 | 0.45% |
| | 20 | -1.1554 | -1.1365 | 1.64% |
| | 50 | -2.8885 | -2.7750 | 3.93% |
| | 100 | -5.7770 | -5.3494 | 7.40% |
| 40 | 5 | -0.2919 | -0.2894 | 0.86% |
| | 20 | -1.1674 | -1.1292 | 3.27% |
| | 50 | -2.9186 | -2.6967 | 7.60% |
| | 100 | -5.8371 | -5.0413 | 13.63% |
| 50 | 5 | -0.2603 | -0.2571 | 1.20% |
| | 20 | -1.0411 | -0.9939 | 4.53% |
| | 50 | -2.6027 | -2.3366 | 10.22% |
| | 100 | -5.2054 | -4.2814 | 17.75% |
| 60 | 5 | -2.2263 | -2.1926 | 1.51% |
| | 20 | -0.8905 | -0.8403 | 5.64% |
| | 50 | -2.2331 | -1.9542 | 11.96% |
| | 100 | -4.4527 | -3.5142 | 21.08% |
| 70 | 5 | -1.8593 | -1.8268 | 1.75% |
| | 20 | -0.7437 | -0.6960 | 6.41% |
| | 50 | -1.8593 | -1.6008 | 13.90% |
| | 100 | -3.7187 | -2.8604 | 23.08% |
| 80 | 5 | -0.1550 | -0.1521 | 1.87% |
| | 20 | -0.6201 | -0.5777 | 6.83% |
| | 50 | -1.5502 | -1.3233 | 14.64% |
| | 100 | -3.1003 | -2.3547 | 24.05% |
| 90 | 5 | -0.1333 | -0.1306 | 2.03% |
| | 20 | -0.5333 | -0.4945 | 7.28% |
| | 50 | -1.3334 | -1.1273 | 15.46% |
| | 100 | -2.6667 | -1.9968 | 25.12% |

In order to observe the data more intuitively, the data of the same mass corresponding to different velocities are extracted and as shown in Fig. 6.

It can be seen from Fig. 6 that both mass and velocity affect the value of inertia influence coefficient. The greater the mass, the greater the value of inertia influence coefficient. The greater the velocity, the greater the value of inertia influence coefficient. The change of mass can significantly change the inertia influence coefficient. It can be seen from Fig. 6 that when the mass of moving load is large, the influence of inertial effect on vibration response can't be neglected regardless of the velocity of the moving load. When the mass of moving load is small, the effect of inertia on vibration response at low speed can be neglected. However, for objects moving at high speed, as shown in the figure, when the mass of moving load is 20 kg and the moving speed is 90 m/s, the coefficient of inertia will exceed 5%, the inertia effect to the vibration response influence can't be neglected in this situation.

### 4.2.2. The Effect of Acceleration on the Coefficient of Inertia Effect

The acceleration analysed in this paper is uniform acceleration. The influence of inertia on vibration response is considered when the moving load performs uniform acceleration motion and uniform deceleration respectively. Considering the uniform acceleration movement with an initial velocity of 10 m/s and the uniform deceleration movement with an initial velocity of 80 m/s, record thelateral displacement of the free end when the moving load moves to the free end. Take acceleration as the only variable, collecting multiple sets of data for analysis. The results of uniform acceleration motion are shown in Table 3.

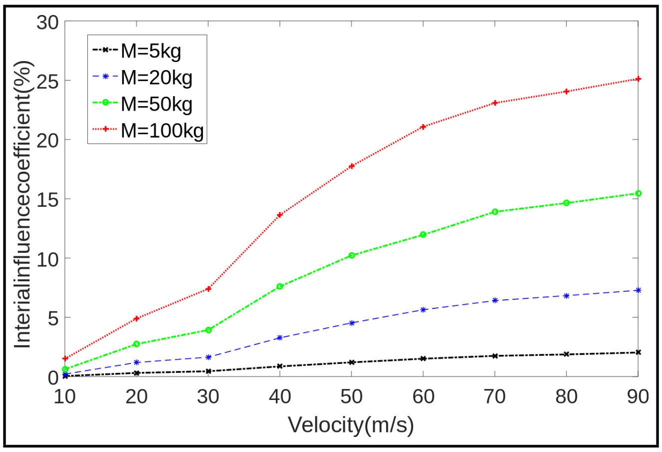


**Figure 6.** Inertial effect coefficients of moving loads with different masses and velocities.

**Table 3.** Response results of moving loads with different masses in the process of acceleration.

| $L_1$= 2 m, $L_2$= 4 m, $L_3$= 2 m, $D_1$= 0.4 m, $d_1$= 0.16 m, $D_2$= 0.2 m, $d_2$= 0.1 m, $D_3$= 0.14 m, $d_3$= 0.04 m | | | | | |
|---|---|---|---|---|---|
| $V$ (m/s) | $M$ (kg) | $A$ (m/s$^2$) | Displacement/ mm: Only considering gravity | Displacement/ mm: Considering inertia effect | Inertial influence coefficient $\lambda$ |
| 10 | 5 | 0 | -0.2790 | -0.2789 | 0.05% |
| | | 10 | -0.2771 | -0.2766 | 0.18% |
| | | 20 | -0.2742 | -0.2733 | 0.33% |
| | | 30 | -0.2677 | -0.2666 | 0.41% |
| | | 40 | -0.2663 | -0.2652 | 0.41% |
| | | 50 | -0.2688 | -0.2676 | 0.45% |
| | | 60 | -2.7230 | -2.7092 | 0.51% |
| | 50 | 0 | -2.7904 | -2.7730 | 0.62% |
| | | 10 | -2.7711 | -2.7226 | 4.85% |
| | | 20 | -2.7422 | -2.6525 | 3.27% |
| | | 30 | -2.6769 | -2.5081 | 6.31% |
| | | 40 | -2.6628 | -2.5635 | 3.73% |
| | | 50 | -2.6881 | -2.5786 | 4.07% |
| | | 60 | -2.7230 | -2.5984 | 4.58% |

The results of uniform deceleration motion are shown in Table 4.

In order to observe the data more intuitively, the data of the same mass corresponding to different accelerations are extracted and shown in Figs. 7 and 8.

Under the condition of uniform acceleration, the inertia influence coefficient increases when the absolute value of acceleration increases. Under the condition of uniform deceleration, as the value of negative acceleration changes, the inertial influence coefficient changes very little.

It can be seen from Figs. 7 and 8 that when the mass of moving load is small, the effect of acceleration on the coefficient of inertia is negligible. In all situations, the coefficient of inertia does not exceed 3%. When the velocity of moving load is small, the coefficient of inertia does not even exceed 1%, and

**Table 4.** Response results of moving loads with different masses in the process of deceleration.

| $L_1$= 2 m, $L_2$= 4 m, $L_3$= 2 m, $D_1$= 0.4 m, $d_1$= 0.16 m, $D_2$= 0.2 m, $d_2$= 0.1 m, $D_3$= 0.14 m, $d_3$= 0.04 m | | | | | |
|---|---|---|---|---|---|
| | | | Displacement/ mm | | Inertial influence coefficient |
| $V$ (m/s) | $M$ (kg) | $A$ (m/s$^2$) | Only considering gravity | Considering inertia effect | $\lambda$ |
| 80 | 5 | 0 | -0.1550 | -0.1521 | 1.87% |
| | | -10 | -0.1561 | -0.1532 | 1.86% |
| | | -20 | -0.1607 | -0.1577 | 1.87% |
| | | -30 | -0.1617 | -0.1587 | 1.86% |
| | | -40 | -0.1626 | -0.1597 | 1.78% |
| | | -50 | -0.1703 | -0.1640 | 3.70% |
| | | -60 | -0.1679 | -0.1648 | 1.85% |
| | 50 | 0 | -1.5502 | -1.3233 | 14.64% |
| | | -10 | -1.5611 | -1.3348 | 14.50% |
| | | -20 | -1.6067 | -1.3710 | 14.67% |
| | | -30 | -1.6166 | -1.3822 | 14.50% |
| | | -40 | -1.6261 | -1.3930 | 14.33% |
| | | -50 | -1.6703 | -1.4288 | 14.46% |
| | | -60 | -1.6786 | -1.4394 | 14.25% |

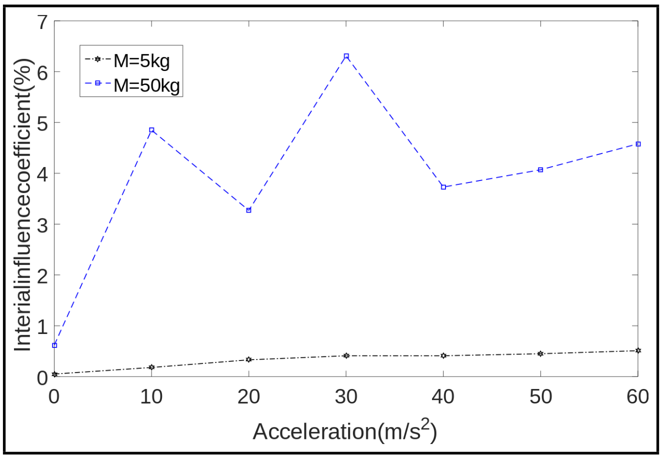


**Figure 7.** Uniform acceleration motion with an initial velocity of 10 m/s.

the effect of inertia on vibration response can be neglected.

When the mass of moving load is large, increasing the positive acceleration can increase the coefficient of inertia from 0.62% to 4.58% when the velocity is 10 m/s. When the velocity is 80 m/s, changing the magnitude of acceleration has almost no effect on the coefficient of inertia, the value is around 14.5%. In this situation, the influence of the coefficient on the vibration response must be considered.

In general, when the mass of moving load is large, the influence of its inertial effect on the vibration response can't be ignored. The coefficient of inertia increases when the velocity of moving load increases. With the increasing of the mass of moving load, the effect is more significant. Acceleration has little effect on inertia influence coefficient in this example, it is only effective at changing the inertia influence coefficient when the mass is large and the velocity is small.

## 5. CONCLUSION

In this paper, taking the three-step cantilever beam as an example, the calculation of natural frequency and the vibration response of moving load considering of inertial effect are carried out. Conclusions are as follows:

1. The mode shape function of the ladder beam is similar to the mode shape function of the constant section beam.

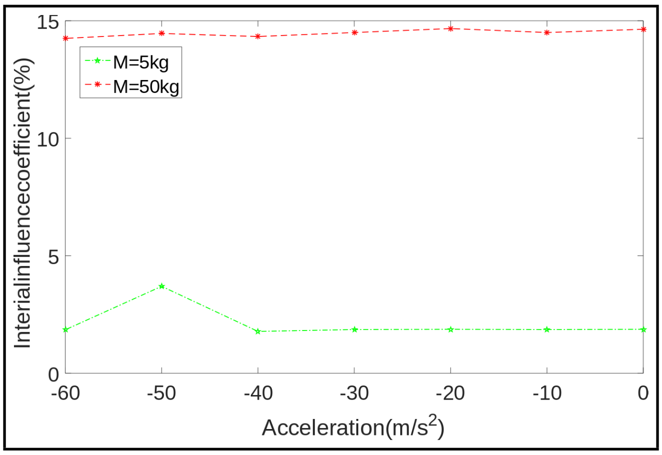


**Figure 8.** Uniform deceleration motion with an initial velocity of 80 m/s.

When the sectional moment of inertia decreases, the image of the mode shape function is steeper.

2. The mass, velocity, and acceleration of the moving load all affect the coefficient of inertia. The greater the mass and speed of the moving load, the greater the coefficient of inertia.

3. When the moving load mass is large and when the effect of inertia on vibration response is negligible when the moving speed and the moving acceleration are relatively small. The influence of inertia on the vibration response can't be neglected when the velocity or acceleration is large. When the moving load mass is small, the effect of the inertia effect on the vibration response can generally be ignored. Although the inertia influence coefficient increases with the increasing of velocity and acceleration, it usually does not exceed 5%. Thus, the mass of the moving load is the key factor to change the coefficient of inertia.

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