Limits

appVersion(4) = "0.99.7822.147"

lim

```
lim    F :=  [ f := F  x := vx  Clear(XLIM)  ans := "" ]                          Initialize
vx→a       δ := str2num(concat("'δ", num2str(x)))
           r(f, x, v) := | r := concat("re(", num2str(x), "):", num2str(f))      Replace functions
                         | tmp := str2num(r)                                      (could equrep or at)
                         | re(v)
           R(f) := | num2str(r(f, x, 0))                                          Convert to string for
                                                                                  better work with try
           if ¬(num2str(d/(d XLIM := x) f) = "0")                                 Check if f depends of x
             if (num2str(a) = "∞") ∨ (num2str(a) = "-∞")
               f := r(f, x, 1/x)                                                  Convert ∞ to x -> 0
             else
               if ¬(num2str(a) = "0")                                             Convert a to x -> 0
                 f := r(f, x, x + a)
               else
                 0
             try
               | ans := R(f)                                                      Try to substitute, then
               | str2num(ans)                                                     f is continue
             on error
               | H(f) := | N := numden(f)
               |         | if ¬lim_LH ∧ try                                       Check L'Hôpital
               |         |        ((R(N_1) = "0") ∧ (R(N_2) = "0")) = 0           conditions,
               |         |      on error                                          lim.LH helps
               |         |        1
               |         |   ans
               |         | else
               |         |   try
               |         |     | ans := R(f)
               |         |     | str2num(ans)
               |         |   on error                                             Try to apply LH rule
               |         |     | for k ∈ [1..MaxI_lim]                            some few times
               |         |     |   | try
               |         |     |   |   ans := R( (d^k/(d XLIM := x)^k N_1) / (d^k/(d XLIM := x)^k N_2) )
               |         |     |   | on error
               |         |     |   |   0
               |         |     |   | if ans = ""
               |         |     |   |   continue
               |         |     |   | else
               |         |     |   |   break
               |         |     | ans
               | if (L := H(f)) = ""                                              Try to apply LH rule to
               |   | if (L := H(ln(f))) = ""                                      f, ln(f) and exp(f)
               |   |   | if (L := H(e^f)) = ""
               |   |   |   | try
               |   |   |   |   L := r(f, x, εl)
               |   |   |   | on error
```

```
|  |  |  |  |  |  Ļ := r(f, x, δ)
|  |  |  |  |  if num2str(IsDefined(Ļ)) = "1"
|  |  |  |  |  |  if |Ļ| > 0.01 εl^(-1)
|  |  |  |  |  |    ∞
|  |  |  |  |  |  else
|  |  |  |  |  |    if |Ļ| < 100 εl
|  |  |  |  |  |      0
|  |  |  |  |  |    else
|  |  |  |  |  |      Ļ
|  |  |  |  |  else
|  |  |  |  |    Ļ
|  |  |  |  else
|  |  |  |    ln(str2num(Ļ))
|  |  |  else
|  |  |    e^(str2num(Ļ))
|  |  else
|  |    str2num(Ļ)
|  else
|    f
```

Try to see if lim is 0 or ∞ numerically

Fails all attemps. return δx as infinitesimal.

LH rule of exp(f) works

LH rule of ln(f) works

LH rule works

f indep of x case

Defaults $\mathrm{MaxI}_{lim} := 8$ $\varepsilon l := 10^{-8}$ $\lim_{LH} := 0$

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lim examples

Extended real number line

SMath rules assumes a > 1 finite number, and have some bugs.

$a + \infty = \infty$ $a - \infty = -\infty$

$a \cdot \infty = \infty$ $\frac{a}{\infty} = 0$

$\infty^{a} = \infty$ $\infty^{-a} = 0$

$a^{\infty} = \infty$ $a^{-\infty} = 0$

But this two cases are wrong:

$0.5^{\infty} = \infty$

$0.5^{-\infty} = 0$

Trivial cases

$\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a} \frac{\sin(x)}{x} = \frac{\sin(a)}{a}$

$\lim_{x \to a} a \cdot x^{2} + x = a \cdot \left(1 + a^{2}\right)$

$\lim_{x \to \infty} 5 \cdot x = \infty$

$\lim_{x \to \infty} 4 \cdot x^{2} - 5 \cdot x = \infty$

$\lim_{x \to \infty} a \cdot x^{2} - b \cdot x = \frac{a - b\ \delta x}{\delta x^{2}}$

where δx->0

Quotient of polinomials

$\lim_{x \to -5} \frac{x^{2} - 25}{x^{2} + 2 \cdot x - 15} = \frac{5}{4}$

$\lim_{x \to 3} \frac{x^{2} - 5 \cdot x + 6}{x - 3} = 1$

$\lim_{x \to 2} \frac{x^{2} + 4 \cdot x - 12}{x^{2} - 2 \cdot x} = 4$

$\lim_{x \to \infty} \frac{5 \cdot x + 3}{3 \cdot x^{2} - 2 \cdot x} = 0$

$\lim_{x \to \infty} \frac{5 \cdot x^{2} + 3}{3 \cdot x^{2} - 2 \cdot x} = \frac{5}{3}$

$\lim_{x \to \infty} \frac{5 \cdot x^{3} + 3}{3 \cdot x^{2} - 2 \cdot x} = \infty$

With another unknow sometimes can get the result, ando sometimes returns the **infinitesimal form**

$$\lim_{x\to 3} \frac{x^3-8\cdot x^2+21\cdot x-18}{2\cdot x^2-12\cdot x+18}=\frac{1}{2}$$

$$\lim_{x\to 3} \frac{x^3-8\cdot x^2+a\cdot x-18}{2\cdot x^2-12\cdot x+b}=\frac{3\cdot(-21+a)}{-18+b}$$

$$\lim_{x\to 0} \frac{2\cdot(x-3)^2}{x}=\infty$$

$$\lim_{x\to 0} \frac{2\cdot(x-3\cdot a)^2}{x}=\frac{2\cdot(\delta x-3\cdot a)^2}{\delta x}$$

where δx -> 0

Infinity and numerical limits

If there are not other variables, and L'Hôpital rule fails, lim try to simplify the expression to **zero** or **infinity** in a numerical **aproximated** way, so, it can result in a wrong value. But I can't get any example of that.

$$\lim_{x\to \infty} \frac{3\cdot x^2-2\cdot x}{5\cdot x+3}=\infty \qquad \lim_{x\to 2} \frac{1}{(x-2)^3}=\infty \qquad \lim_{x\to 0} \frac{2\cdot(x-3)^2}{x}=\infty \qquad \lim_{x\to \infty} \frac{x^2-2\cdot x}{x+3}=\infty$$

$$\lim_{x\to \infty} \frac{x}{\ln(x)}=\infty \qquad \lim_{x\to 0} \frac{\ln(x)}{x}=\infty \qquad \lim_{x\to 0} \frac{\ln(a\cdot x)}{a\cdot x}=\frac{\ln(a\ \delta x)}{a\ \delta x}$$

Potential, exponential and logarithmic functions

$$\lim_{x\to 0} \sqrt[-b\cdot x]{1+a\cdot x}=\frac{1}{\sqrt[b]{\mathbf{e}}^{a}} \qquad \lim_{x\to \infty} \sqrt[\frac{b}{x}]{\frac{x}{x+a}}=\frac{1}{\sqrt[b]{\mathbf{e}}^{a}}$$

$$\lim_{x\to 0} \sqrt[\frac{x}{b}]{1+a\cdot x}=\mathbf{e}^{a\cdot b} \qquad \lim_{x\to \infty} \left(1+\frac{a}{x}\right)^{b\cdot x}=\mathbf{e}^{a\cdot b}$$

$$\lim_{x\to 0} \sqrt[a\cdot x]{x+1}=\sqrt[a]{\mathbf{e}} \qquad \lim_{x\to \infty} \sqrt[\frac{a}{x}]{\frac{x+1}{x}}=\sqrt[a]{\mathbf{e}}$$

$$\lim_{x\to 0} \sqrt[\frac{x}{a}]{x+1}=\mathbf{e}^{a} \qquad \lim_{x\to \infty} \left(\frac{x+1}{x}\right)^{a\cdot x}=\mathbf{e}^{a}$$

$$\lim_{x\to 0} \frac{\mathbf{e}^{a\cdot x}-1}{x}=a \qquad \lim_{x\to \infty} x\cdot\left(\sqrt[\frac{x}{a}]{\mathbf{e}}-1\right)=a$$

$$\lim_{x\to 0} \frac{\ln(1+a\cdot x)}{x}=a \qquad \lim_{x\to \infty} x\cdot\ln\left(1+\frac{a}{x}\right)\cdot x=-\frac{a^2}{2}$$

Wrong due a bug in ln, see below.

$$\lim_{x\to \infty} \frac{\ln(x)}{x^2}=0 \qquad \lim_{x\to \infty} \frac{x^2}{\ln(x)}=\infty \qquad \lim_{x\to \infty} \frac{\ln\left(\frac{1}{x}\right)}{x^2}=0 \qquad \lim_{x\to \infty} x^2\cdot\ln\left(\frac{1}{x}\right)=\infty$$

Trigonometric functions

$$\lim_{x\to 0} \frac{\sin(a\cdot x)}{b\cdot x}=\frac{a}{b} \qquad \lim_{x\to 0} \frac{1-\cos(a\cdot x)}{b\cdot x^2}=\frac{a^2}{2\cdot b} \qquad \lim_{x\to 0} \frac{\tan(a\cdot x)}{b\cdot x}=\frac{a}{b}$$

$$\lim_{x \to 0} \frac{\sin(x^2)}{x^2} = 1 \qquad \lim_{x \to 0} \frac{1-\cos(x^2)}{x^2} = 0 \qquad \lim_{x \to 0} \frac{1-\cos(a \cdot x)}{b \cdot x} = 0$$

$$\lim_{x \to 0} x^2 \cdot \cos\left(\frac{1}{x}\right) = 0 \qquad \lim_{x \to 0} x \cdot \sin(x) = 0 \qquad \lim_{x \to \infty} x \cdot \sin(x) = \infty$$

Gallery

$$\lim_{x \to 0} \frac{1}{x^3} \cdot \left(\left(\frac{2+\cos(x)}{3}\right)^x - 1\right) = -\frac{1}{6} \qquad \lim_{x \to \pi} \frac{\sin(2 \cdot x)}{\sin(x)} = -2$$

$$\lim_{x \to 0} \frac{\tan(x) - x}{x^3} = \frac{1}{3} \qquad \lim_{x \to 0} \left(1 + \sin(5 \cdot x)\right)^{\frac{1}{\tan(x)}} = \mathbf{e}^5$$

$$\lim_{x \to \infty} \frac{\sqrt{x^3 - x^2 + 3 \cdot x}}{\sqrt{x^3} - \sqrt{x^2} + \sqrt{3 \cdot x}} = 1.0001$$

Numeric approximation. Can control the precission with εl.

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = 1$$

Notice that for this example L'Hôpital rule fails, but the code can find the correct value.

Indeterminate forms

All error messages are ok, except for 0/0:

$\frac{0}{0} = ▮$ **lastError** = "Division by zero."

$$\lim_{x \to 0} \frac{\sin(x) - a \cdot x}{x^3} = -\frac{1}{6}$$

$\infty - \infty = ▮$ **lastError** = "Uncertainty."

$$\lim_{x \to \infty} 4 \cdot x - \ln(x^2) = \infty$$

$\frac{\infty}{\infty} = ▮$ **lastError** = "Uncertainty."

$$\lim_{x \to \infty} \frac{\ln(x)}{x^3} = 0$$

$\infty^0 = ▮$ **lastError** = "Uncertainty."

$$\lim_{x \to \infty} \sqrt[x]{x} = 1$$

Not a big news

$0 \cdot \infty = ▮$ **lastError** = "Uncertainty."

$$\lim_{x \to 0} x \cdot \ln(x) = 0$$

$1^\infty = ▮$ **lastError** = "Uncertainty."

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \mathbf{e}$$

$0^0 = ▮$ **lastError** = "Uncertainty."

$$\lim_{x \to 0} x^{x \cdot \ln(x)} = 1$$

Derivatives

This is more or less obvious just becaus lim is based on diff.

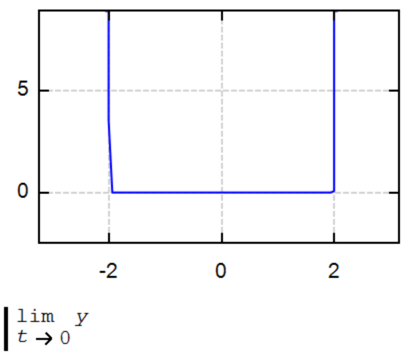
$$f(x) := x \cdot \cos(a \cdot x) \qquad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = -a \cdot \sin(a \cdot x) \cdot x + \cos(a \cdot x)$$

$f(x) := \sqrt{a \cdot x + b}$ $\qquad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{a}{2 \cdot \sqrt{b + a \cdot x}}$

Using the infinitesimal δx

We can study the behaviour of the limit when it returns δ's

$y := \sqrt[t]{\left| t^2 + \frac{a}{2} \right|}$ $\qquad \lim_{t \to 0} y = \sqrt[\delta t]{\left| \frac{a + 2\,\delta t^2}{2} \right|}$ $\qquad \delta t := 10^{-3}$



From the plot and this case $\qquad \lim_{t \to 0} y \Big|_{a = 2} = 1$

we conclude: $\qquad \lim_{t \to 0} y = \begin{cases} 0 & \text{if} \quad a < 2 \\ 1 & \text{if} \quad a = 2 \\ \infty & \text{otherwise} \end{cases}$

Known issues

- If a function gives an error with diff, it will be given an error with lim too.

  $xo := 4.5$ $\qquad \frac{d}{d\,xo} \text{Gamma}(xo) = ■$ $\qquad$ **lastError** = "xo - not defined."

- $\lim_{x \to 4} \frac{x - \sqrt{3 \cdot x + 4}}{4 - x} = -\frac{10485761}{16777216}$

  SMath can't simplify for check L'Hôpital rules, and return an approximate value. We help with

  $\lim_{LH} := 1$

  $\lim_{x \to 4} \frac{x - \sqrt{3 \cdot x + 4}}{4 - x} = -\frac{-3 + 2 \cdot \sqrt{16}}{2 \cdot \sqrt{16}}$

  Problem here is that $\sqrt{16}$ isn't simplify to 4

  $\lim_{LH} := 0$

- $y := \sqrt[-x]{1 + a \cdot \left(e^{-x} - 1 \right)}$ $\qquad \lim_{x \to 0} y = \frac{1}{e^{2 - a}}$

  Wrong. Correct value is e^a

  The bug is here: $\ln(y) = -\frac{\ln\left(e^x + a \cdot \left(-e^x + 1 \right) \right) + x}{x}$

  correct value: $\ln(y) = -\frac{\ln\left(1 + a \cdot \left(e^{-x} - 1 \right) \right)}{x}$

- The code can't handle very well complex expressions or some other expressions, returning numerical approximations instead exact values

  $u := (1 + x)^{\frac{1}{x}}$ $\qquad v := \frac{x^2 - 12}{x^2 - 2}$ $\qquad z := u + i \cdot v$

  $\lim_{x \to 0} z = \frac{612102644706258 + 1351079888211150 \cdot i}{225179981368525}$ $\qquad$ even $\lim_{x \to 0} u = e$ $\qquad \lim_{x \to 0} v = 6$

  $\lim_{x \to 3} \frac{\sin(x)}{x} = \frac{10592466928549}{225179981368525}$ $\qquad$ even $\lim_{x \to a} \frac{\sin(x)}{x} = \frac{\sin(a)}{a}$

  But at least you can evaluate both numerically $\qquad \lim_{x \to 3} \frac{\sin(x)}{x} = 0.047$ $\qquad \lim_{x \to 0} z = 2.7183 + 6 \cdot i$

- There are an option for control the numerical answers, εl value. For example, here the code can try to get a numeric answer, but there are an (not visible for the user) overflow error.

$$\lim_{x \to \infty} x \cdot \mathbf{e}^{-x} = \frac{1}{\delta x \cdot \sqrt[\delta x]{\mathbf{e}}} \qquad \lim_{x \to 0} x \cdot \sqrt[x]{\mathbf{e}} = \delta x \cdot \sqrt[\delta x]{\mathbf{e}}$$

$$\varepsilon l := 10^{-1} \qquad \lim_{x \to \infty} x \cdot \mathbf{e}^{-x} = 0 \qquad \lim_{x \to 0} x \cdot \sqrt[x]{\mathbf{e}} = \infty \qquad \varepsilon l := 10^{-8}$$

- The code do not recognice directions, which can introduce some issues. For example, let

$$y_1 := \frac{1}{3 + \sqrt[x]{2}} \qquad \lim_{x \to 0} y_1 = \frac{1}{3 + \sqrt[\delta x]{2}}$$

Both are function only of x, for get a closed result change εl value

$$y_2 := \frac{1 + \sqrt[x]{2}}{3 + \sqrt[x]{2}} \qquad \lim_{x \to 0} y_2 = \frac{1 + \sqrt[\delta x]{2}}{3 + \sqrt[\delta x]{2}}$$

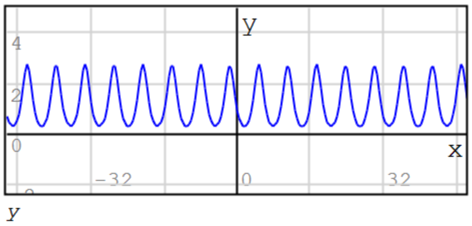
$\varepsilon l := 10^{-3}$ $\lim_{x \to 0} y_1 = 0$ but $\lim_{x \to 0^+} y_1 = 0$ and $\lim_{x \to 0^-} y_1 = \frac{1}{3}$

$\lim_{x \to 0} y_2 = 1$ $\lim_{x \to 0^+} y_2 = 1$ and $\lim_{x \to 0^-} y_2 = \frac{1}{3}$ $\varepsilon l := 10^{-8}$

- The code can returns a value even the function oscilates.

$$y := \frac{x + \cos(x) \cdot \sin(x)}{\mathbf{e}^{\sin(x)} \cdot (x + \cos(x) \cdot \sin(x))}$$

$$\lim_{x \to \infty} y = 0.3939$$



Alvaro