7 A Bit of Rocket Science

7.1 Nine Days of One Year

(based on a text by V. F. Ochkov) In this study, issues related to the motion of carrier rockets, both one- and two-stage, are considered. The study delves into Tsiolkovsky's formula and the selection of an optimal location for a cosmodrome. Differential equations are solved both numerically and analytically. Additionally, the study explores the use of mathematical formulas in the construction of monuments. The mathematical aspects include the solution of algebraic and differential equations.

The study also delves into manual calculations carried out by characters in a feature film. The physical aspects discussed encompass rocket motion, Tsiolkovsky's formula, Einstein's theory of relativity, acceleration due to gravity, and Earth rotation. In the realm of information technology (IT), the study explores the solution of equations and the operator of choice.

We start with a dialogue from the Soviet feature film "Nine Days of One Year" between actors Yevgeniy Yevstigneyev (theoretical physicist Nikolai Ivanovich — left in Figure 7.1) and Mikhail Kazakov (romantic physicist Valery Ivanovich — right). The film, by the way, can be obtained as DVD with german subtitles at Petershop.com

Figure 7.1: Scene of "Nine Days of One Year", Michael Romm, Mosfilm (1961), also available as DVD with German subtitles.

—Tell me, Valery Ivanovich, how deep are you going to penetrate the depths of our galaxy?

—At first only to the depth of 500 light years.

—At what speed?

—Close to the speed of light.

—The weight of your ship?

—One hundred thousand tonnes.

—The fuel?

—The most advanced.

—Now we will calculate how much fuel you will need. Be kind, napkin. ...

Valery Ivanovich leaves the table and comes back after some time :

—Nikolai Ivanovich, have you finished your calculations?

—Yes please. For a space ship of one hundred thousand tonnes weight at a speed close to the speed of light, in order to fly around a part of the galaxy in a reasonable time for human life, you need ten to the power of twenty two of the most modern extra-fuel. As reference: Our planet weighs a little less. Happy journey!

The dialogue is, of course, naive, but imagine that Nikolai Ivanovich does not have a napkin in his hands, but ... a "tablet" (tablet computer) with SMath Studio installed, and he makes calculations on it.

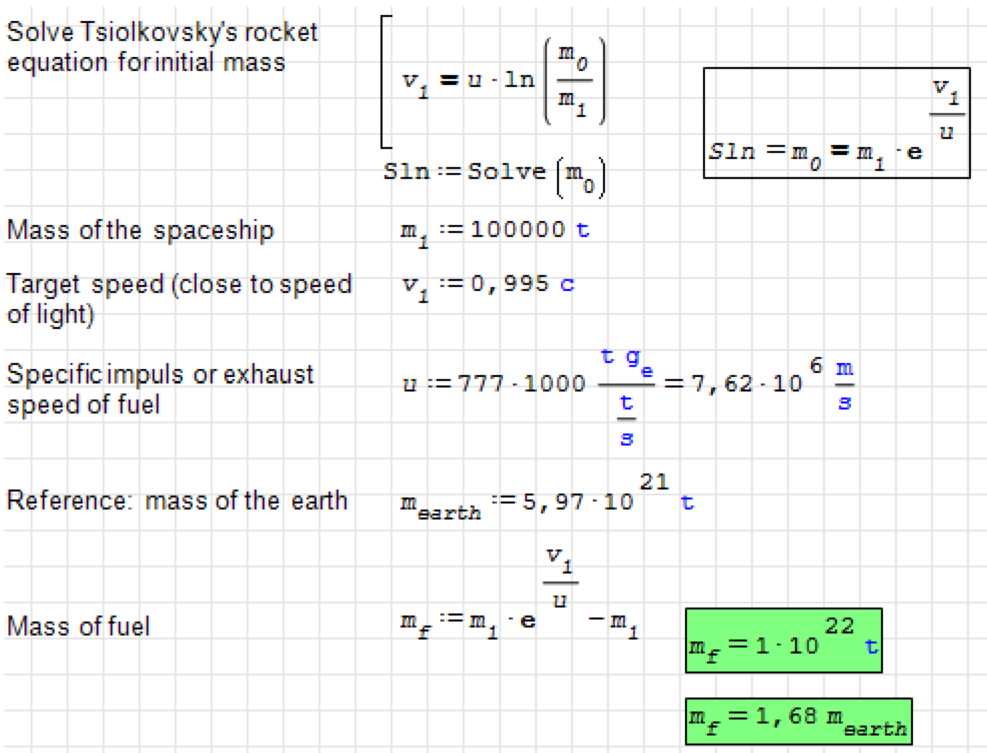

Figure 7.2: Calculations on the napkin in "Nine days of One Year" (Serviette.sm)

In the calculation in Figure 7.2, Tsiolkovsky's formula was used (by the way, Tsiolkovsky himself is mentioned in the movie): The final velocity of the rocket v_1 is proportional to the natural logarithm of the ratio of the starting mass of the rocket m_0 to its final mass m_1. The difference $m_0 - m_1$ is the mass of fuel and oxidant (hereinafter — simply fuel, as in the film). The proportionality factor in this equation (the value of u) is the specific impulse. It is equal to the ratio of the thrust of the engine F to the fuel mass consumption per time μ (these quantities will be used in further calculations). The film doesn't say anything about this quantity, so we selected the value such as to meet the stated mass of fuel. The value of the specific impulse is also the velocity of the outflow of gases from the rocket nozzle (meters per second). The theoretical value of this velocity for nuclear rocket engines can exceed 70 km/s. The exhaust velocity for an electric motor can reach 140 km/s. Therefore, it is quite possible to dream about the "three sevens".

The Tsiolkovsky equation in Figure 7.2 is solved with respect to the variable m_0. If the final speed of the rocket is close to the speed of light (we assumed that this speed is 0.995 of the speed of light c), then the mass of fuel actually exceeds the mass of the Earth m_{earth}. But our calculation is, of course, very rough. In addition, it does not take into account the important fact that at speeds close to the speed of light, Tsiolkovsky's formula does not work. Here it will be necessary to depart from the laws of classical mechanics and resort to the theory of relativity of Einstein. Nikolai Ivanovich says "Our planet weighs a little less". It seems that in fact, the planet weighs much less than the required fuel.

Questions:

- Can the crew survive a flight with the assumptions from Figure 7.2? Assume that the maximum acceptable acceleration is 10 g.
- How long is the time from launch to engine cut-off (Brennschluss)? Note that the expected distance of travelling of 500 light years at approximate speed of light would imply a duration of at least 500 years. We don't know why this is called a "reasonable time for human life" by Nikolai Ivanovich.
- What is the fatal consequence of burning all the fuel at the beginning of the journey?

7.2 Drilling Down

We now have a closer look at a single stage rocket. The mass at launch consists of the payload P, the mass of the carrier m and the mass of the fuel M. The engine consumes the fuel at the rate μ (mass per time) and ejects it with the average speed u. This speed depends on the technology of the engine, e.g. the temperature of the exhaust gases. Thus, the total mass ramps down linearly from launch mass $m_0 = P + m + M$ to $m_{end} = P + m$ at engine cut-off. Cut off is assumed to happen when no fuel is left over, therefore cut-off time is $t_{end} = m/\mu$.

The engine trust is the product of exhaust speed times the burnt mass per time unit: $F = \mu u$. It is constant under the given assumptions. The resulting acceleration of the rocket is $\ddot{h} = F/m(t) - g$. It increases due to decreasing mass up to $F/(P + m)$.

Figure 7.3 shows the mathematical model along with analytical and numerical solutions for speed and altitude over time (assuming vertical launch). For the derivation of the analytical solution see e.g. (Gummert/Reckling, 1987).

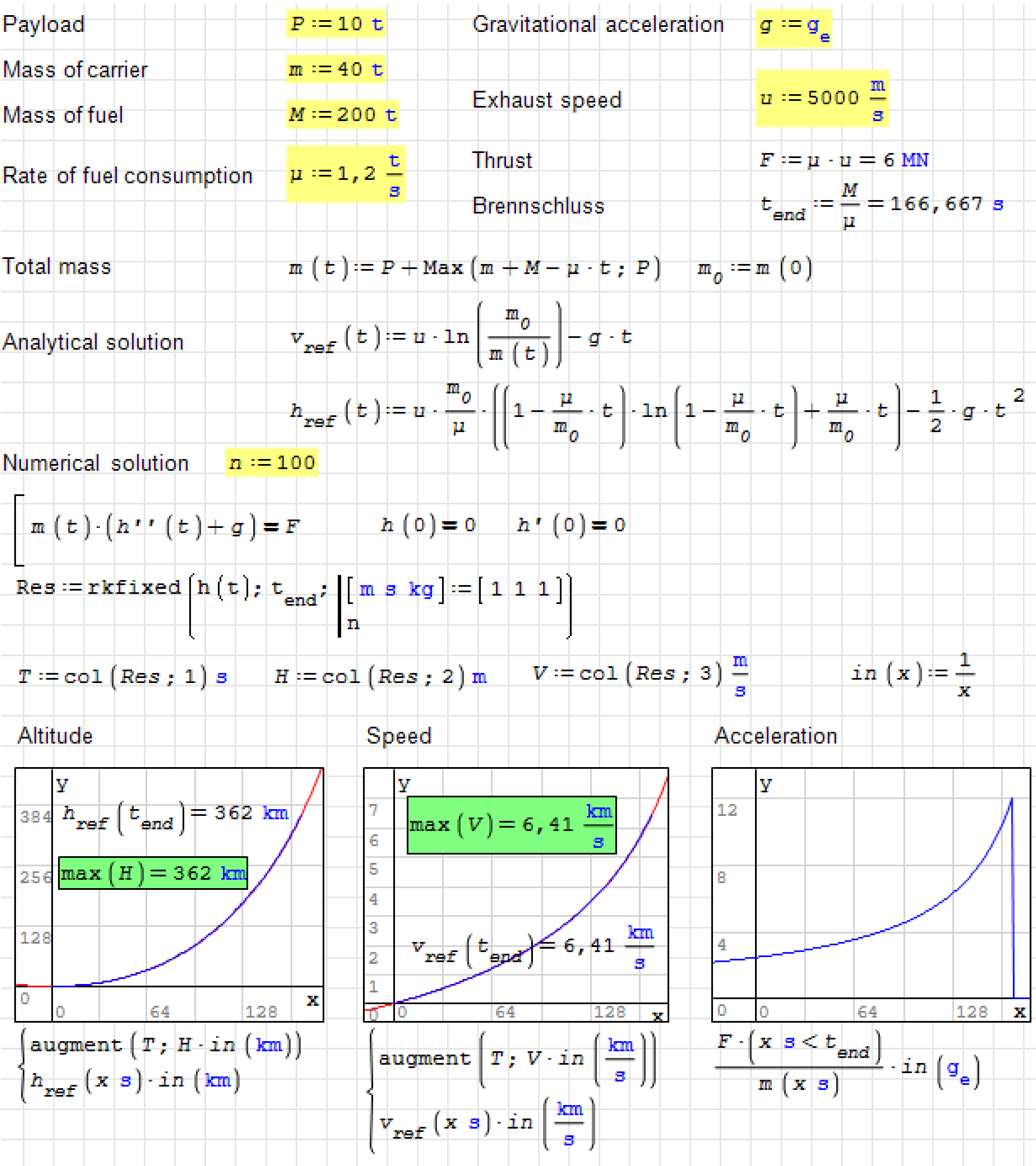

Figure 7.3: Model for a single-stage rocket. (rocket.sm)

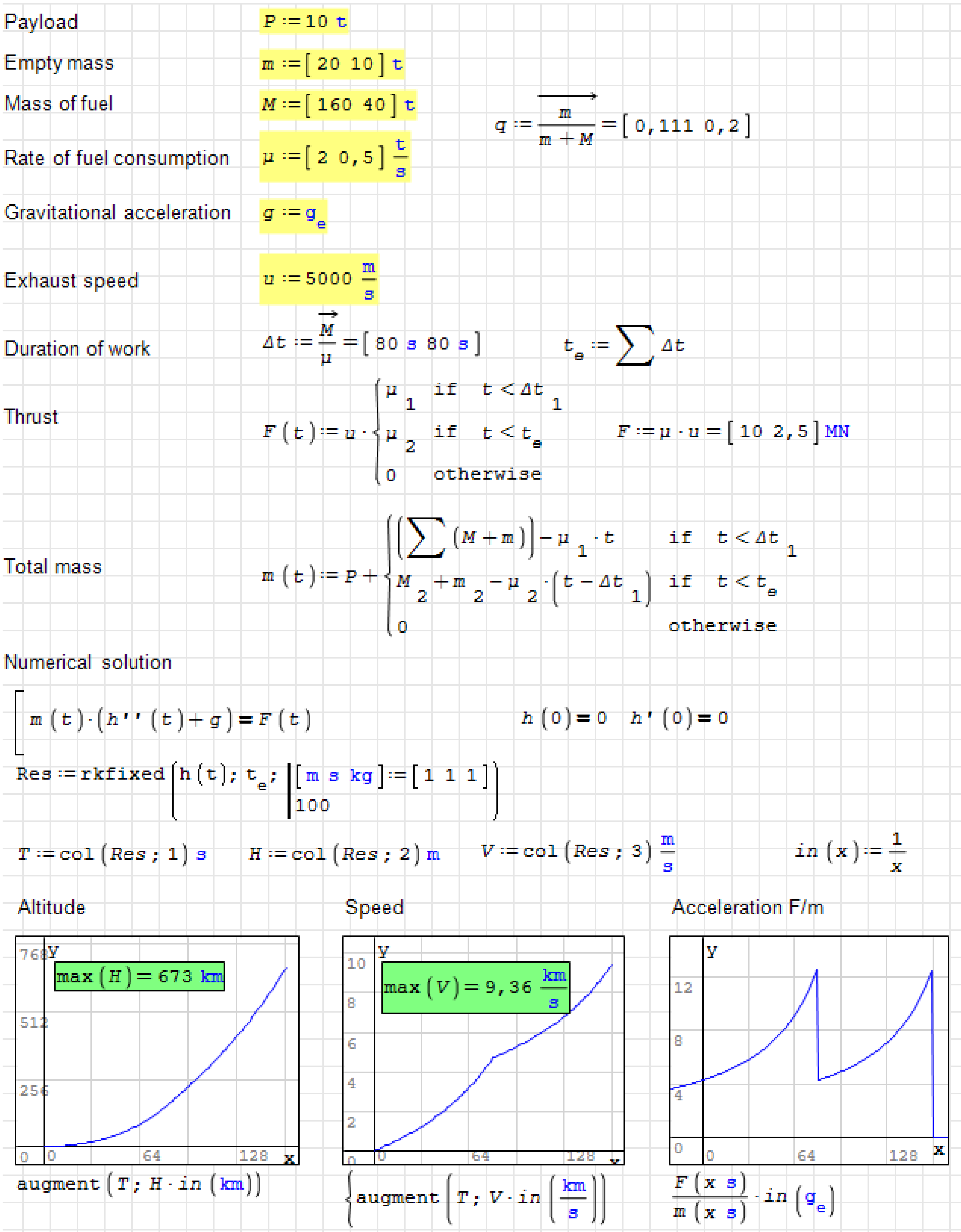

Figure 7.4: Model for a two-stage rocket (rocket2.sm)

Figure 7.4 shows model and solution for a two-stage rocket. In order to have a fair comparison, we assume this rocket to use the same amount of fuel, the same payload, the same maximum acceleration and the same engine technology (exhaust velocity). Fuel mass M, carrier mass m and rate of fuel consumption μ are set separately for two stages.

We also assume that the rate of fuel consumption is proportional to the mass of the stage and that the ratio of fuel mass to empty mass is equal for both carrier stages. The example shows that the flight speed at engine cut-off is significantly higher than what can be reached with a single-stage rocket.

Questions:

- What would be the final speed at engine cut-off for a single-stage and a two-stage-rocket if F/m is limited to $10\,g$?
- Modify the two-stage-model such that the mass split is controlled by a single parameter α with $m_1 = \alpha m$ and $m_2 = (1 - \alpha)$. Apply this accordingly to M and adjust μ such that the acceleration is limited to $10\,g$.
- Try to find an optimal mass split for a two-stage rocket.
- Find out how fast you can get with a three-stage rocket with the same amount of fuel, structural mass and exhaust velocity (and with the $10\,g$ limit applied).
- How do the performance parameters (final speed and altitude) change if the reduction of gravity with increasing distance from Earth is taken into account by $g(h) = g\,r_E^2/(r_E + h)^2$ with $r_E = 6371\,\text{km}$?
- What distance will the spaceship of Valery Ivanovich have travelled at engine cut-off?